XVAs
Preliminary version

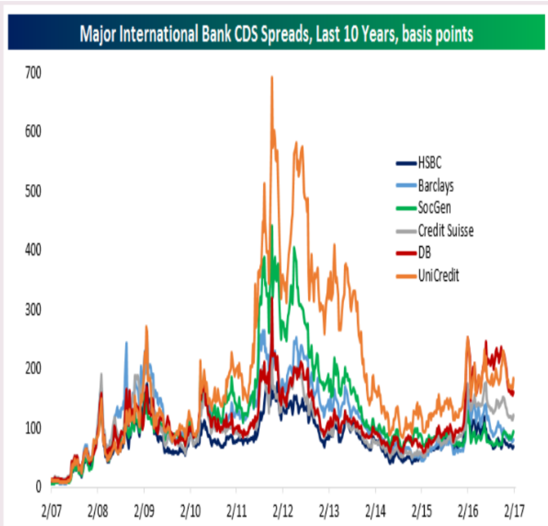
Cyril Bénézet
ENSIIE, LaMME
cyril.benezet@ensiie.fr

March 4, 2021

Chapter 1

Introduction

1.1 Historical introduction



During the 2008-2009 crisis, the corporate credit spreads became much higher, reflecting that the probability of default of corporations was much more significant at that time. While the pricing of financial derivatives already incorporated counterparty default risk using a CVA add-on, it was said afterwards that "roughly two-thirds of losses attributed to counterparty credit risk were due to CVA losses and only about one-third were due to actual defaults".

What does it mean ?

When entering a deal with a client, consisting in future exchanges of random cash-flows, the dealer bank on one side computes the *clean value* of the deal, without consideration of counterparty risk and its implication, and it computes on the other side rebates, or value adjustments, to take into account these additional risks. The price charged to the client is thus the difference of the clean value and the value adjustments: the bank pays (if positive) MtM − FTP, where MtM is the clean value and FTP is the *funds transfer*

price (FTP), the sum of all the rebates that are included in the entry price.
Intuitively, the CVA is computed at time $t = 0$ as the expected cost of the future expenses due to the default of the counterparty. However, the credit rating of the client can evolve through time, and this expected cost can vary with time. If, as during the crisis, the client's rating becomes worse, then the expectation of future costs due to counterparty defaults CVA_t becomes higher. The bank needs to possess, at each time t, the quantity CVA_t in order to face these expected costs. CVA_0 is paid by the client at the deal inception and reserved, but the bank itself has to adapt the amount in the reserve capital account so that it matches CVA_t for all t. This results in losses for the bank when CVA_t becomes higher and higher due to the worsening of the client's rating.

After the crisis, regulators launched a major banking reform in order to secure the financial system by *collateralization and capital requirements rules.*
However, to post collateral, the bank often needs to borrow money from external funders at a borrowing spread rate (higher than the risk free rate). In addition, shareholders are expected to put capital at risk so that the bank can cover exceptionnal losses (i.e. beyond expected losses), and they in turn expect in this regard a risk premium for their immobilized capital at risk.
The reform did not take into account these costs of capital and of funding. Their quantification by the banks, under reglementary market incompletness that we will explain, gave birth to other value adjustments as an unintended consequence of the banking reform.

Definition 1.1.1. XVAs *are, at a deal inception,* pricing add-ons *(*Value Adjustments*) representing the cost for the different risks induced by counterparty defaultability and its funding and capital implications due to collateralization and capital requirement.*

Here, X is a *catch-all letter* meant to be replaced:

- C for credit,
- D for debt,
- F for funding,
- M for margin,
- K for capital.

We will often merge MVA and FVA to save a letter, but the distinction will be explained below.

As we saw earlier, it is important for the bank to compute the add-ons at the deal inception, but it is also (and perhaps more) important to take into account the whole XVA processes, as the variations of these quantities through time generate losses for the bank. The XVAs are *accounting entries* in the balance sheet of the bank, and XVAs profit and losses are reported and can be found in the results of a company.

Another example to show that understanding the dynamics of the XVA processes is important: in 2014, JP Morgan recorded a $1.5 billion (!) FVA loss.

What needs to be understood:

- What are the XVAs?
- How to compute the XVAs at the trade level, in order to fix a price for a deal?
- How to understand, model and take into account the variations of the XVA processes, responsible for losses?
- How to compute the XVA add-ons and to simulate the XVA processes?

1.2 How does a dealer bank work? Connection with the balance sheet

1.2.1 How does a dealer bank work?

We describe here how a dealer bank works when engaging in a bilateral portfolio with a client.

First, the bank is divided into three different desks:

- the *clean desk* is responsible for the deals with the clients, and has to hedge the deals, focusing on market risks and forgetting the counterparty risks, by the work of the other desks.
- the *CA desks*, divided into the CVA desk and the FVA desk, has to deal with so called *contra-assets*. The contra-assets (which are liabilities to the bank) are the cash-flows the bank has to face due to counterparty risk and its funding implications. The CVA desk specifically deals with the counterparty cash-flows, while the FVA desk is the Treasury and deals with the cash-flows related to the funding policy of the bank and the collateralization requirements.
- the KVA desk is the Management is in charge of the shareholder capital at risk and their remuneration, i.e. the dividend release policy and the capital implications of the counterparty risk.

Assume that a client wants to enter a bilateral trade with the dealer bank. Assume further that all the bank accounts are empty for simplicity. The deal is contracted following these steps:

1. the bank computes the clean value MtM of the deal, and the various rebates $\mathrm{FTP} = \mathrm{CA} + \mathrm{KVA}$ (we assume here that $\mathrm{MtM}, \mathrm{FTP} > 0$), where CA is the expected value of the contra-assets at $t = 0$ and KVA the capital value adjustment, expected cost to remunerate the shareholders at $t = 0$.

2. the client pays FTP to the bank, and the money is posted into different accounts.

   - The counterparty and funding value adjustments are posted in the *reserve capital account* is an asset which will be used to cover the losses of the CA desks, i.e. realizations of the contra-assets.
   - The capital value adjustment is posted in the *risk margin account*, which is added to the shareholder capital at risk to obtain the capital at risk: this is the capital that the management can use to deal with exceptionnal losses and to release dividends to remunerate the shareholders.

3. The Treasury of the bank borrows money from external funders, depending on which accounts are usable as a funding source, so that it owns MtM. This amount MtM is posted in the *clean margin account.* For example, if only the reserve capital account is a funding source, the Treasury has to borrow $(\mathrm{MtM}-\mathrm{CA})^+$ if necessary. We will often assume that the reserve capital account is the only funding source, but in practice the capital at risk is also a funding source. Last, let us mention that we did not discuss collateral between the client and the bank, which of course induce more funding effects.

4. The clean margin account can be seen (if the amount contained is positive) as the funding debt put at disposition by the CA desk to the clean desk. Here it contains MtM, exactly what is needed by the clean desk to pay the client at time $t = 0$ to enter the deal.

Note the following relations at $t = 0$:

- The amount in the reserve capital account is denoted by RC, and $\mathrm{RC} = \mathrm{CA}$ at $t = 0$, paid by the client to the CA desks.
- The amount in the risk margin account is denoted by RM, and $\mathrm{RM} = \mathrm{KVA}$ at $t = 0$, paid by the client to the KVa desk.
- The amount in the clean margin account is denoted by CM, and $\mathrm{CM} = \mathrm{MtM}$ at $t = 0$, paid by the CA desk to the clean desk.

The clean margin account is collateral exchanged between the CA desks and the clean desk.
At time $t = 0$, we saw that it contains MtM to fund its debt.
In fact, at each time $t \geq 0$, this account is assumed marked-to-model to the clean value of the deal (or, more generally, of the portfolio), to guarantee the clean desk against counterparty defaults. More precisely, at the time of default τ:

1. the clean margin account (or more precisely the net amount relative to the cash-flows impacted by the default) property is transferred from the CA desk to the clean desk. It contains $\mathrm{MtM}_{\tau-}$, the value of the deal just before the default.

2. During the *liquidation period*, i.e. the period during which the default is resolved, the counterparty in default sells assets in order to pay the cash-flows up to the recovery rate, and the non-defaulted party continues to pay the cash-flows until the end of the liquidation period.
   If the client defaulted, the CVA desk pays the amount unpaid by the client, i.e. the promised cash-flows minus the recovered part.
   If the bank defaulted, the clean desk pays to the client the recovered part of the promised cash flows, and the promised cash-flows minus the recovered part to the CVA desk.
   In any case, the clean desk recieves or pays the full cash-flows it should have recieved or paid, as if nobody had defaulted, until the liquidation time, when the deal is terminated.
   In addition, during this period, the CVA desk still provides to the clean desk the MtM fluctuations of the deal. The risk that the missing cash-flows and the fluctuations of the MtM are important during the liquidation period is called the *gap risk*.

To summarize, if $\mathcal{P}^c$ is the process representing the cumulative promised cash-flows between the client and the clean desk, and if P^c is the price process for these cash-flows (i.e. $P_t^c = \mathbb{E}_t\left[\mathcal{P}_T^c - \mathcal{P}_t^c\right]$ under the risk-neutral measure) the clean desk will actually recieve the process $\mathcal{P}$ with dynamics:

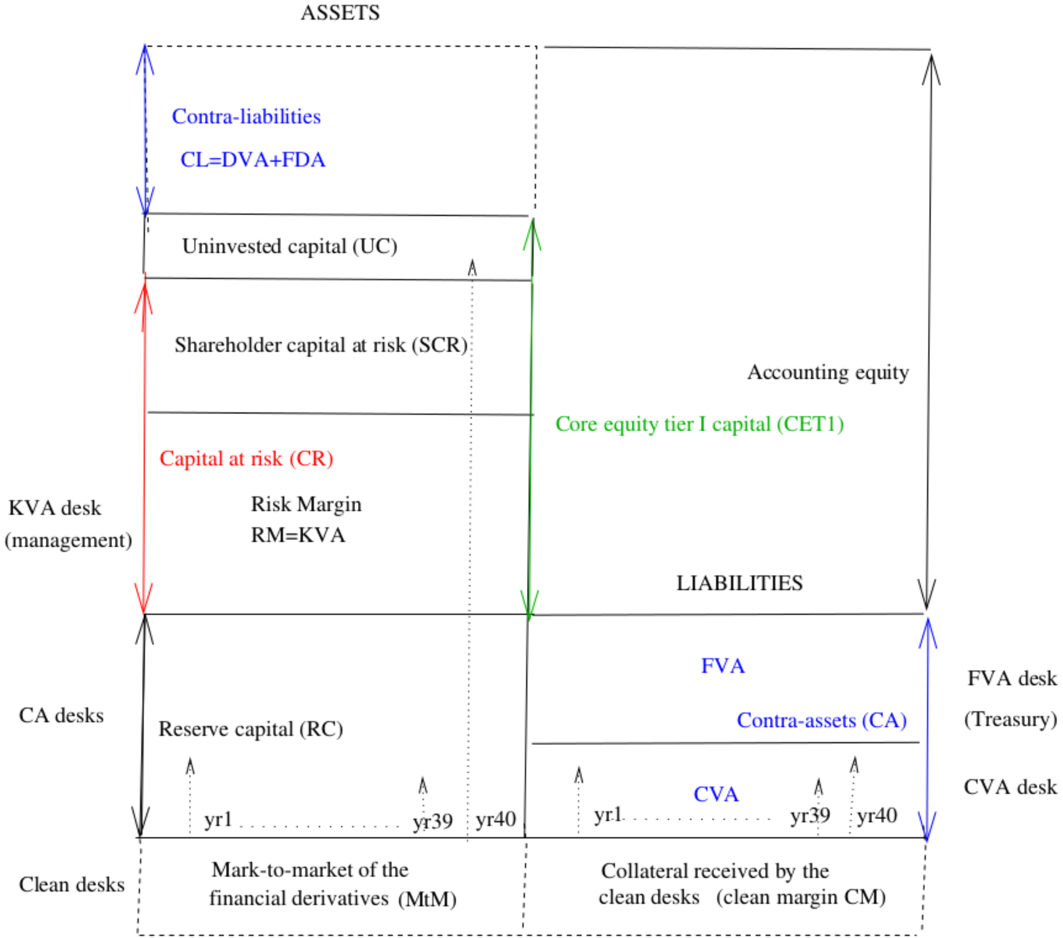
$$d\mathcal{P}_t = 1_{\{t<\tau\}} d\mathcal{P}_t^c + P_{\tau-}^c \delta_\tau(dt) + 1_{\{\tau \le t \le \tau+\delta\}}(d\mathcal{P}_t^c + dP_t^c).$$

Indeed, the first term corresponds to the cash-flows exchanged as promised before default. The second term is the clean margin account becoming the property of the clean desk at the default, and the third term represents the cash-flows exchanged between the clean desk, the CVA desk and the client during the liquidation period $[\tau, \tau+\delta]$, as described above. The amount MtM corresponds to the value of the deals involving the client, so after the default it becomes 0, which coincides avec the fact that the clean margin account becomes empty after the default, i.e.

$$d\mathrm{MtM}_t = dP_t^c 1_{\{t<\tau\}} - P_{\tau-}^c \delta_\tau(dt),$$

meaning that $\mathrm{MtM}_t = P_t^c 1_{\{t<\tau\}}$.

1.2.2 Connection with the Balance Sheet



We can connect the previous functioning of a dealer bank to its balance sheet. The bottom line is the balance sheet of the clean desk. Its assets are the promised cash-flows exchanged between the different clients of the bank, with expected value MtM, and its liability is the clean margin account, which is property of the CA desks before the default of the bank. As we explained above, this account is assumed to be mark-to-model (i.e. match, at all times, the model value of) the mark-to-market, so that the assets equals the liabilities at all time: CM = MtM.

The second line refers to the CA desks: the liabilities are the contra-assets, so the liabilities value is the expected value of these cash-flows. The corresponding assets are in the reserve capital: at time $t = 0$ (or at each deal inception), the amount is provided by the client, but the variations between deals are assumed by the bank, so that at all time, the marked-to-model equality RC = CA is satisfied.

The third line refers to the KVA desk: the asset is the capital at risk, which is loss-absorbing, meaning that it is reserved to cover exceptional losses which the reserve capital account is not enough for. There is also some *uninvested capital* UC (which is essentially an adjustment variable for us), so the full shareholder capital is $\mathrm{SHC} = \mathrm{SCR} + \mathrm{UC}$, and CET1 is the *core equity tier I capital*, defined as $\mathrm{CET1} = \mathrm{SHC}+\mathrm{RM} = \mathrm{SCR}+\mathrm{UC}+\mathrm{KVA}$, which represents the financial strength of the bank from a regulatory and structural point of view. The risk margin account is also marked-to-model to the KVA so that $\mathrm{RM} = \mathrm{KVA}$.

Last, the *contra-liabilities* are the future cash-flows that are gains for the bank due to its own defaultability. We denote by CL the expected value of the contra-liabilities. We will describe them in the next section. Since they are only value after the bank defaults, they are naturally not part of the core equity tier I capital. After the default of the dealer bank, the bondholders take the decisions instead of the shareholders: in particular, the contra-liabilities are value to the bondholders and not to the shareholders, and are not part of the pre-default balance sheet.

To summarize, all the accounts are supposed to be **marked-to-model**, so we have the following balance conditions, assumed to be satisfied at all times:

$$\begin{aligned} \mathrm{CM} &= \mathrm{MtM}, \\ \mathrm{RC} &= \mathrm{CA}, \\ \mathrm{RM} &= \mathrm{KVA}. \end{aligned}$$

Remark 1.2.1. *Regarding the bank default time, we will assume that this is a* ***totally unpredictabe time*** τ *calibrated to the bank CDS spread curve. This curve is the most reliable information about credit data, anticipations of market participants, government interventions, future recapitalization. In particular, we allow* recapitalization *and* CET1 *can be negative without triggering default.*

1.2.3 Collateralization

We have seen that there is collateral exchanged between the CA desks and the clean desk using the clean margin account, but collateral is also exchanged between the bank and the client. In both cases, the goal is to guarantee some value to their counterparties in case of default. The collateral is split into two parts: *variation margin* (VM) and *initial margin* (IM).

The goal of variation margining is to post cash tracking the mark-to-market value of the portfolio. To do so, there are so called variation margin call times, where both parties readjust their variation margin. It typically consists of re-hypothecable cash, meaning that, for example, recieved VM can be used for other funding purposes. We assume that it is remunerated risk-free.

However, even if the variation margin were (as we assume in the clean margin account), at all times, exactly equal to the mark-to-market value of the portfolio, there would be some *gap risk*, as described above: during the liquidation period, the CVA desk has to pay for missing cash-flows and fluctuations of the MtM. The IM is posted to partially cover this risk. Contrarily to the VM, IM is non-fungible across deals, so the IM account's value cannot be used for other funding purposes.

To raise collateral, the bank can use cash posted in re-hypothecable accounts. But it also often has to borrow money from their external funders, who lend unsecured to the dealer bank, meaning that they suffer a loss-given-default in case of default of the bank. Since this lending is unsecured, the external funders charge an interest rate equal to the risk-free rate plus a spread, to take into account the bank's default probability over time. This external funding thus induces costs for the bank.

1.3 Presentation of the various XVAs

We explained that XVAs were introduced to account for, anticipate and monetize the expected losses for the dealer bank coming from the counterparty default risk and its funding and capital implications. We describe more precisely here each value adjustment.

1.3.1 CVA

CVA stands for *Credit Value Adjustment*. As explained above, the CVA desk has to deal with the "counterparty contra-assets" (liabilities): it has to post collateral to the clean desk in the clean margin account, which becomes the clean desk property in case of default. Moreover, it has to compensate for the missing cash-flows and mark-to-market fluctuations during the liquidation period. CVA_t is the expected cost for the CVA desk of these losses, due to the defaultability of the counterparty.

Notice that when a counterparty defaults, it impacts only the set of deals between this specific client and the bank. Since the CVA compensates only for positive cash-flows from the client to the bank (and not for negative cash-flows, from the bank to the client as the bank has not defaulted), we observe that the CVA depends only on the positive part of the cash-flows: the CVA is the price process for a *derivative*. Moreover, due to netting effects (i.e. cash-flows from the bank to the client in some deals, compensated by cash-flows from the client to the bank in other ones), it is not optimal (from a "minimality" point of view) to compute one CVA for each deal: one has rather to compute one CVA for each client, considering the whole portfolio of deals between the client and the bank.

To summarize: the bank has one CVA for each counterparty, which is a derivative much more complicated than any deal: the underlying asset for the CVA is the whole net value of the dealer's portfolio cash-flows involving this counterparty.

From a numerical and practical point of view, it would be easier to have one CVA for each deal: at every new deal, the bank only consider the new contract and computes the expected cost for the counterparty cash-flows generated by the deal. But we saw that the bank is better off computing one CVA for each counterparty, which immediately makes the CVA computations more involved: when contracting a new deal with one client, one has to

1. compute the CVA of the portfolio without this new deal CVA^{old} at the deal inception (or observe this quantity which should be in the CVA part of the reserve capital account dedicated to this counterparty),
2. compute the CVA of the portfolio including the new deal CVA^{new} at the deal inception,
3. charge to the client the difference $\text{CVA}^{\text{new}} - \text{CVA}^{\text{new}}$, so the new CVA part of the RC account dedicated to this counterparty contains the correct CVA, including the new deal.

We compute here *incremental* CVAs.

To emphasize on the difficulty to compute the CVA for one counterparty, note that the joint law between the counterparty's future probability of default and the dealer's exposure (over all deals involving this counterparty) has to be taken into account, and that the CVA is a derivative which payoff is a non-linear function of the underlying. One could of course simplify the problem by assuming independence between the default of the counterparty and the bank's exposure towards it. But one would neglect important effects: there can be situations with a positive dependence between the two, meaning that the probability of default is high when the exposure to the counterparty is high. This situation is referred to as *wrong-way risk*. The opposite is called *right-way risk*.

Last, we give a concrete example on how consequent is the task: at the time of its failure, Lehman had about 1.5 million derivatives transactions with 8.000 different counterparties. It thus had to compute 8.000 CVAs. (**Reuters 2008**)

1.3.2 DVA

DVA stands for *Debt Value Adjustment*, which is the symmetric companion of the CVA. As explained above, the bank expects counterparty profits due to its own defaultability, as some cash-flows from the bank to its counterparties will not be entirely paid after the bank's default: these are the contra-liabilities (assets) associated consequences of the bank's default on the deals. More precisely, the CVA desk expects profits due to the bank defaultability, as the clean desk will pay to them rather to the client some cash-flows. DVA_t is the expected value for these profits for the CVA desk.

Considering DVA into the dealers balance sheet led to seemingly paradoxical statements: in the first quarter of 2009, Citigroup reported a positive mark to market (meaning a gain) due to its worsened credit quality: if the price paid by the bank at $t = 0$ is $\text{MtM} - \text{CVA} + \text{DVA}$, and the DVA becomes higher at time $t > 0$, then, all other things being equal, the t-value is $\text{MtM} - \text{CVA} + \text{DVA}_t$ which is higher than before. This led to a debate in 2011.

We explain in the Funds Transfer price section what is the result of this debate, and why DVA (and more generally contra-liabilities) should not be taken into account in the pricing of a new deal.

1.3.3 FVA, MVA and FDA

FVA (resp. *MVA*) stands for *Funding Value Adjustment* (resp. *Margin Value Adjustment*) and FDA stands for *Funding Debt Adjustment*. As explained above, the Treasury has to bear the cost of "funding contra-assets" (liabilities): due to collateralization requirements, the Treasury will have to borrow money to external funders. Since this borrowing is unsecured, the interest rate includes a spread computed using the bank probability of default, so borrowing from external funders comes with a price. There are also funding contra-liabilities (assets): when the bank defaults, the external funders suffer a loss-given-default: the bank pays only a fraction (defined by the recovery rate) of what it has borrowed. Hence the bank expects funding profits due to its own defaultability.

FVA_t is the expected cost at time t for funding variation margin, while MVA_t is the expected cost at time t for funding initial margin.

FDA_t is the expected value that the external funders lose due to the bank's default, or the expected gains for the bank due to its own defaultability. As for the DVA, we refer to the Funds Transfer Price section to discuss if this should be priced or not (it should not).

These value adjustments depend upon the risky funding policy of the bank, the accounts that are funding sources or not, the decomposition of collateral amounts into VM, which is re-hypothecable, and IM, which is segregated. Consequently, the MVA, which is only concerned with the cost induced by IM, can be computed at the level of each deal, the FVA should be computed at the level of the whole portfolio of the bank to achieve minimality: if it was computed at the trade or counterparty level, important netting effects would be forgotten, resulting in higher FVA amounts.

This is a challenging numerical task, in fact more challenging that the computation of CVA, as one has to consider the joint law of all risk factors underlying every deal and counterparty in the portfolio. When entering a new deal, the FVA must be computed with an incremental approach as described in the case of the CVA, but for the FVA, the bank has to consider its whole portfolio with and without the new trade.

Last, let's mention that we will discuss below connections with the Modigliani and Miller theorem.

1.3.4 KVA

KVA stands for *Capital Value Adjustment*: KVA_t is the expected cost for the bank to remunerate the shareholders for their capital at risk.

Since the others XVAs are defined as the expected cost for cash-flows triggered by counterparty and its funding implications, they are not enough to cover *exceptional losses*, especially considering that hedging counterparty and funding risks is not practical. The bank shareholders then have to put capital at risk on top of the reserve capital account. They expect this risk to be remunerated at a hurdle rate by the bank. KVA is thus the value for the bank having to remunerate the shareholders.

In the framework we develop, the KVA will be priced at the deal inception as a risk-premium, on top of the other value adjustments. Let us mention that the KVA debate is not really settled today.

From a numerical perspective, the capital at risk that must be reserved by the shareholders is computed as a *risk measure* (value at risk or expected shortfall) on the loss of the bank over a year. The loss of the bank depends itself on the contra-assets realization and the dynamics of other XVAs. Thus one needs to compute every other XVAs before this one, and obviously incremental KVA is at the level of the whole portfolio.

1.3.5 Funds Transfer Price

Using all the previous add-ons directly, the price adjustment should be (merging FVA and MVA to save a letter)

$$\mathrm{FTP} = \mathrm{CVA} - \mathrm{DVA} + \mathrm{FVA} - \mathrm{FDA} + \mathrm{KVA}.$$

We will observe in addition that FVA = FDA. We then obtain

$$\mathrm{FTP} = \mathrm{CVA} - \mathrm{DVA} + \mathrm{KVA},$$

which is the fair and symmetrical adjustment between two counterparties of equal bargaining power, on top of which the KVA comes to remunerate the shareholders for their capital at risk.

However, we now argue that we should not include the contra-liabilities expected value CL = DVA + FDA in the entry price.

Recall that these cash-flows are only profits to the bank after its default, so they are destinated to the bondholders and not to the shareholders who control the bank at deal inception. Since these are future profits, it should initially cost to the bank, but the initial payment is done before default by the shareholders. There is no reason why shareholders would pay for profits destinated to the bondholders, so they are not willing to include it in the price.
The only situation in which the shareholders would still pay for the contra-liabilities would be that they are able to monetize before default these cash-flows, by contracting a further deal at $t = 0$, delivering the contra-liabilities as a payment to a third party. The shareholders would then obtain at $t = 0$ the amount $\mathrm{DVA} + \mathrm{FDA}$, but the bondholders would have to pass their profits to the third party after the default of the bank. Hopefully, the bondholders are protected by laws of *pari-passu* type, and this kind of deal is forbidden. This reveals market incompletness due to regulation constraints, underlying the fact that the interests of the shareholders and of the bondholders are not aligned.
This shows that bank is not willing to price contra-liabilities at the deal inception. Last, we also recall that clients are *price-takers*: they do not decide for the price and do not have the bargaining power to argue that their missing cash-flows due to the bank's defaultability should be priced at their advantage.

This leads to the formula

$$\mathrm{FTP} = \mathrm{CVA} + \mathrm{FVA} + \mathrm{KVA},$$

which is the sum of the *contra-assets* $\mathrm{CA} = \mathrm{CVA} + \mathrm{FVA}$ and of the risk premium KVA.

The *contra-liabilities* $\mathrm{CL} = \mathrm{DVA} + \mathrm{FDA}$ are still (post-default) accounting entries, assets for the bondholders in case of bank's default.

1.3.6 Connection with the Modigliani–Miller theorem

The Modigliani and Miller (1958) theorem states, under conservation of total wealth and market completness, that the fair valuation of counterparty risk is independent of its funding policy, and that funding and capital structure policies of a firm are irrelevant to the profitability of its investment decisions. The connections between the XVAs paradigm (especially regarding funding contra-assets and value adjustments) and this theoem his led to a FVA debate around 2013.
In what we described, total wealth is conserved but, the complete market hypothesis is relaxed, as the bank is not allowed to hedge post-bank default cash-flows.
If it were, contra-liabilities expected value would be incorporated in entry prices, and the formula $\mathrm{FTP} = \mathrm{CVA} - \mathrm{DVA} + \mathrm{FVA} - \mathrm{FDA} + \mathrm{KVA} = \mathrm{CVA} - \mathrm{DVA} + \mathrm{KVA}$ would be valid, which is the sum of the fair valuation of counterparty risk $\mathrm{FV} = \mathrm{CVA} - \mathrm{DVA} + \mathrm{FVA} - \mathrm{FDA}$ and the risk premium KVA, so the second conclusion of the theorem would be valid as $\mathrm{FV} = \mathrm{CVA} - \mathrm{DVA}$ is independent of the funding policy.
Due to incompletness, the second conclusion of the theorem is not valid anymore, as the

formula $\text{FTP} = \text{CVA} + \text{FVA} + \text{KVA}$ now holds, and the funding policy appears in FVA. However, the fair valuation of counterparty risk $\text{FV} = \text{CVA} - \text{DVA} + \text{FVA} - \text{FDA} = \text{CVA} - \text{DVA}$ is still independent of the funding policy, according to the first conclusion of the theorem, as $\text{FVA} = \text{FDA}$.

1.4 XVA spirit

The XVAs are pricing add-ons to take into account counterparty risk and its funding and capital implications, coming collateralization and capital at risk, inducing additional costs to the bank. More than that, they are accounting entries varying through time and which need to be monitored as their variations are potential sources of important losses for the bank. They also help to design a dividend release policy.

1.4.1 Satisfying the regulatory constraints

The point of view developed in these notes for the XVAs construction underlies the fact that a dealer bank has to satisfy to regulatory constraints.

First, the **Volcker rule** forbids a bank to do proprietary trading, meaning that it should not make profits entering in deals with counterparties. In the following, we will assume that the bank is **perfectly hedged** against market risk: all cash-flows that are positive to the bank due to the deals are compensated by a negative cash-flow going to the hedging market, and *vice-versa*.

In particular, this also allows to focus our conclusions on the XVAs: counterparty, funding and capital losses. If the bank were not perfectly hedged, one would only need to add a term in the loss process of the bank.

Secondly, the bank has to respect laws of ***pari-passu*** **type**. This means that the shareholders cannot contract deals that would trigger value away from bondholders during the default resolution. Indeed, after the bank's default, the incoming cash-flows (contra-liabilities) only benefit to the bondholders, and monetizing these cash-flows before default is forbidden.

As explained above, these laws imply structural market incompleteness in the market that the shareholders have to take into account.

Still, we will discuss the valuation impact of the (theoretical) inclusion of deals hedging the contra-liabilities.

1.4.2 A conservative approach

Last, we will assume that the CA desks do no hedge on the counterparty and funding risks. If we included some XVA hedge, this would change nothing to the qualitative conclusions of the theory, but the smaller loss that would be computed would imply a smaller economic capital and KVA.

1.4.3 Possibiity to go into run-off

At each new deal, we will compute the XVAs as if the bank was not going to contract any further deal, in the limiting case of a "run-off" portfolio. The point is to avoid "Ponzi schemes", where the bank needs to enter in more and more deals in order to pay previously contracted ones. The incremental XVA approach we develop will be constructed in order for the bank to have, at any time, the possibility to go into run-off, still taking into account the shareholders interest: their wealth process SHC will be a submartingale with constant hurdle rate.

Chapter 2

One period model

We consider a price-maker bank contracting a deal with a price-taker client in a one-period model. The bank has no previous deal and contracts this deal at time $t = 0$, with maturity $t = 1$. The portfolio is in *run-off*, meaning that no other deal is contracted.

If the bank were in a counterparty risk-free linear market, it would pay MtM (or recieve from the client if it is negative) to the client to enter the deal, which is computed (see below) as the expected value of the (discounted) future cash-flows. However, pricing rebates CA = CVA + FVA are computed beforehand to take into account counterparty risk and its funding implications, and the corrected price is MtM − CA. On top of that, shareholders have to put capital at risk to satisfy to regulatory constraints, they thus deserve a risk premium, which is also incorporated in the price, which is eventually MtM − CA − KVA.

2.1 Probabilistic setup

We consider a probabilistic setup $(\Omega, \mathcal{A}, \mathbb{Q}^\star)$ where the sigma-algebra $\mathcal{A}$ encompasses all the available information, in particular the defaults of the bank and of each of its clients.

The probability measure $\mathbb{Q}$ will be used both for pricing and XVA computations, and for risk measures computations, so $\mathbb{Q}$ is neither the physical measure nor the risk-neutral one. One can think of $\mathbb{Q}^\star$ as the unique probability measure on $(\Omega, \mathcal{A})$ that coincides

- with a given risk-neutral measure on the financial sigma algebra $\mathcal{F}$
- with the physical probability conditional on the financial sigma algebra,

see Artzner et al. (2020).

We define the following cash-flows:

- $\mathcal{P}^c$ are the *promised* cash-flows from the client to the bank,
- $\mathcal{C}$ are the *counterparty* cash-flows from the CVA desk to the clean desk.

- $\mathcal{F}$ are the *risky funding* cash-flows from the bank to its external funders,
- $\mathcal{H}$ are the *hedging* cash-flows from the bank to the hedging market.

The firm valuation of counterparty risk (both coming from the client's default and the bank's default) is the expected cost of the associated cash-flows:

$$\text{FV} = \mathbb{E}^\star(\mathcal{C} + \mathcal{F}).$$

We also define *survival indicators*

- J is the one for the bank,
- J_1 is the one for the client.

Remark 2.1.1. *$J = 1$ means that the bank has not defaulted at time $t = 1$, whereas $J = 0$ indicates that the bank has indeed defaulted at $t = 1$. The same interpretation holds for J_1 and the client.*

We define $\gamma := \mathbb{Q}^\star(J = 0)$ the $\mathbb{Q}^\star$-probability that the bank defaults.

2.2 Financial assumptions

The following hypothesis are in force.

- The *Volcker rule* holds, meaning in particular that the bank is perfectly hedged against market risk.
- The laws of *pari-passu type* hold, as described in the first chapter, meaning in particular that after the default of the bank, the incoming flows are paid to the bondholders and not to the shareholders. These laws also forbid the bank to hedge itself against its own jump-to-default cash-flows: if it was the case, the bank would monetize these cash-flows at time $t = 0$ and it would prevent the bondholders to obtain them, thus breaking the laws of *pari-passu* type.
- When the bank defaults, the property of the (residual amount on the) reserve capital and of the risk margin accounts are transferred from the shareholders to the bondholders.
- The bank does not implement any hedge for the client default risk: this could in principle be handled by single name credit default swaps, but they are illiquid. This is also a conservative hypothesis, including XVA hedge would only reduce the loss of the bank.
- External funding is fairly priced, meaning that $\mathbb{E}^\star(\mathcal{F}) = 0$,
- The bank has zero recovery to its external funder and to its client. The client has also zero recovery to the bank.

- The risk-free rate is $r = 0$ (or the risk-free asset is taken as *numéraire*).
- Collateral is exchanged between the CA desk and the clean desk through the clean margin account, as described in the first chapter.
- No collateral is exchanged between the clean desk and its client.
- We ignore here the possibility of using capital at risk for funding purposes. The reserve capital is the only funding source (other than external funding).
- The risk margin to remunerate the shareholders for providing the capital at risk is loss-abosrbing.

Under these hypothesis, assuming that the bank has no available cash, to enter into a deal, the bank:

- Charges the client CA and KVA, respectively posted in the reserve capital account and the risk margin account.
- Borrows $(\text{MtM} - \text{CA})^+$ from the external funders, or lends $(\text{MtM} - \text{CA})^-$ risk-free.
- Since we assume that the only funding source is reserve capital, the bank has $\text{CA} + (\text{MtM} - \text{CA})^+ - (\text{MtM} - \text{CA})^- = \text{CA} + \text{MtM} - \text{CA} = \text{MtM}$, which is deposited in the clean margin account.
- The clean desk uses this MtM in the clean margin account to pay the client.

Moreover, by the Volcker rule, the mark-to-market valuation of the deal must be computed in order to be able to construct a perfect hedge. In our linear market, we set

$$P^c = \mathbb{E}^\star \left[\mathcal{P}^c\right],$$

which will also equal MtM, the amount in the clean margin account at $t = 0$. This way, there exists a replication strategy, meaning that $\mathcal{H} = \mathcal{P} - \text{MtM}$. In particular, the Mark-to-Market of the deal at maturity $t = 1$ is zero.

2.3 The clean desk cash-flows $\mathcal{P}$ and the counterparty cash-flows $\mathcal{C}$

We compute here the cash-flows $\mathcal{P}$ that actually recieve the clean desk, together with the counterparty cash-flows $\mathcal{C}$.
Recall that $\mathcal{P}^c$ are the promised cash-flows from the client to the bank. Because of counterparty default and of the work of the CVA desk, the clean desk recieves cash-flows $\mathcal{P}$ which may differ from $\mathcal{P}^c$, see the Introduction. In particular, while the cash-flows $\mathcal{P}^c$ are promised by the client, cash-flows $\mathcal{P}$, which are actually recieved by the clean desk, are coming from two sources: the client, and the CVA desk (through the clean margin account which becomes the property of the clean desk as soon as one party defaults, and

by compensating for the missing cash-flows in the liquidation period).
In the notations introduced above, $\mathcal{P} = \mathcal{C} + (\mathcal{P} - \mathcal{C})$ where $\mathcal{C}$ are the cash-flows between the clean desk and the CVA desk, and $\mathcal{P} - \mathcal{C}$ are the cash-flows between the clean desk and the client.

In our elementary static setup, we show that

Proposition 2.3.1. *We have*

$$
\begin{aligned}
\mathcal{P} &= \mathcal{P}^c, \\
\mathcal{P} - \mathcal{C} &= J_1\mathcal{P}^+ - J\mathcal{P}^-, \\
\mathcal{C} &= (1 - J_1)\mathcal{P}^+ - (1 - J)\mathcal{P}^-,
\end{aligned}
$$

In particular,

$$
\begin{aligned}
\mathcal{C}^\circ &= J(1 - J_1)\mathcal{P}^+, \\
\mathcal{C}^\bullet &= (1 - J)(\mathcal{P}^- - (1 - J_1)\mathcal{P}^+).
\end{aligned}
$$

Proof. We use the following decomposition of any random variable $\mathcal{Y}$ along the partition $\{J = J_1 = 1\}, \{J = 1, J_1 = 0\}, \{J = 0, J_1 = 1\}, \{J = J_1 = 0\}$, noticing that $1_{\{J=1\}} = J1$, $1_{\{J_1=1\}} = J_1$, $1_{\{J=0\}} = 1 - J$ and $1_{\{J_1=0\}} = 1 - J_1$,

$$
\mathcal{Y} = JJ_1\mathcal{Y} + J(1 - J_1)\mathcal{Y} + (1 - J)J_1\mathcal{Y} + (1 - J)(1 - J_1)\mathcal{Y}.
$$

If $\mathcal{Y}$ is a (net sum of) cash-flow(s), the first term corresponds to the cash-flows exchanged when no default occurs. The second (resp. third) term correspond to the situation where the client (resp. bank) only defaults, while the last term correspond to the situation where both the client and the bank default.

Since $\mathcal{P} - \mathcal{C}$ represents the effective cash-flows from the client to the bank, we have the four following situations

- $J = J_1 = 1$: neither the bank nor the client have defaulted, the clean desks recieves from the client the promised cash-flows, and no cash-flow is exchanged between the CVA and the clean desks:

$$
\begin{aligned}
JJ_1\mathcal{P} &= JJ_1\mathcal{P}^c, \\
JJ_1(\mathcal{P} - \mathcal{C}) &= JJ_1\mathcal{P}^c, \\
JJ_1\mathcal{C} &= 0.
\end{aligned}
$$

- $J = 1, J_1 = 0$: the client has defaulted while the bank has not. As a default occured, the clean margin account, which contains at $t = 0 = 1^-$ the amount MtM, becomes the property of the clean desk. Then, at $t = 1$, the client does not pay the clean desk which does not recieve $(\mathcal{P}^c)^+$ from them, but the clean desk pays $(\mathcal{P}^c)^-$ to the client. The CVA desk compensates for the missing cash-flows and MtM variations, thus pays $(\mathcal{P}^c)^+$ to the clean desk and pays $-\text{MtM}$ (i.e. the clean desk pays MtM to the CVA desk) as the MtM goes from MtM at $t = 0$ to 0 at $t = 1$. Thus

$$\begin{aligned} J(1-J_1)\mathcal{P} &= J(1-J_1)\left(\text{MtM} + (\mathcal{P}^c)^+ - \text{MtM} - (\mathcal{P}^c)^-\right) = J(1-J_1)\mathcal{P}^c, \\ J(1-J_1)\mathcal{C} &= J(1-J_1)\left(\text{MtM} + (\mathcal{P}^c)^+ - \text{MtM}\right) = J(1-J_1)\left(\mathcal{P}^c\right)^+, \\ J(1-J_1)(\mathcal{P}-\mathcal{C}) &= -J(1-J_1)\left(\mathcal{P}^c\right)^- . \end{aligned}$$

as $\text{MtM}+(\mathcal{P}^c)^+-\text{MtM}$ is paid by the CVA desk and $-(\mathcal{P}^c)^-$ by the client (meaning that $(\mathcal{P}^c)^-$ is recieved by the client from the clean desk).

- $J = 0, J_1 = 1$: the bank has defaulted while the client has not. As a default occured, the clean margin account, which contains at $t = 0 = 1^-$ the amount MtM, becomes the property of the clean desk. Then, at $t = 1$, the client pays the clean desk which recieves $(\mathcal{P}^c)^+$, but the clean desk does not pay $(\mathcal{P}^c)^-$ to the client. The CVA desk compensates for the missing cash-flows and MtM variations, thus recieves $(\mathcal{P}^c)^-$ and $-\text{MtM}$ from the clean desk. Thus

$$\begin{aligned} (1-J)J_1\mathcal{P} &= (1-J)J_1\left(\text{MtM} - (\mathcal{P}^c)^- - \text{MtM} + (\mathcal{P}^c)^+\right) = (1-J)J_1\mathcal{P}^c, \\ (1-J)J_1\mathcal{C} &= (1-J)J_1\left(\text{MtM} - (\mathcal{P}^c)^- - \text{MtM}\right) = -(1-J)J_1\left(\mathcal{P}^c\right)^-, \\ (1-J)J_1(\mathcal{P}-\mathcal{C}) &= (1-J)J_1\left(\mathcal{P}^c\right)^+ . \end{aligned}$$

- $J = J_1 = 0$: both the bank and the client have defaulted, and there are no cash-flow between the client and the bank. As a default occured, the clean margin account, which contains at $t = 0 = 1^-$ the amount MtM, becomes the property of the clean desk. At $t = 1$, the CVA desk pays $\mathcal{P}^c$ to the clean desk and recieves MtM from the clean desk. Thus

$$\begin{aligned} (1-J)(1-J_1)\mathcal{P} &= (1-J)(1-J_1)\left(\text{MtM} + \mathcal{P}^c - \text{MtM}\right) = (1-J)(1-J_1)\mathcal{P}^c, \\ (1-J)(1-J_1)\mathcal{C} &= (1-J)(1-J_1)\left(\text{MtM} + \mathcal{P}^c - \text{MtM}\right) = (1-J)(1-J_1)\mathcal{P}^c, \\ (1-J)(1-J_1)(\mathcal{P}-\mathcal{C}) &= 0. \end{aligned}$$

We then have

$$\begin{aligned} \mathcal{P} &= JJ_1\mathcal{P} + J(1-J_1)\mathcal{P} + (1-J)J_1\mathcal{P} + (1-J)(1-J_1)\mathcal{P} \\ &= JJ_1\mathcal{P}^c + J(1-J_1)\mathcal{P}^c + (1-J)J_1\mathcal{P}^c + (1-J)(1-J_1)\mathcal{P}^c \\ &= \mathcal{P}^c, \end{aligned}$$

and, using that $\mathcal{P} = \mathcal{P}^+ - \mathcal{P}^-$,

$$\begin{aligned} \mathcal{C} &= JJ_1\mathcal{C} + J(1-J_1)\mathcal{C} + (1-J)J_1\mathcal{C} + (1-J)(1-J_1)\mathcal{C} \\ &= J(1-J_1)\mathcal{P}^+ - (1-J)J_1\mathcal{P}^- + (1-J)(1-J_1)\mathcal{P} \\ &= (1-J_1)\mathcal{P}^+ - (1-J)\mathcal{P}^-, \end{aligned}$$

and eventually

$$\begin{aligned}\mathcal{P} - \mathcal{C} &= \mathcal{P} - (1 - J_1)\mathcal{P}^+ + (1 - J)\mathcal{P}^- \\ &= \mathcal{P}^+ - \mathcal{P}^- - \mathcal{P}^+ + J_1\mathcal{P}^+ + \mathcal{P}^- - J\mathcal{P}^- \\ &= J_1\mathcal{P}^+ - J\mathcal{P}^-.\end{aligned}$$

We thus obtain the result. □

2.4 The funding cash-flows $\mathcal{F}$

In this section, we show that borrowing to the external funders is fairly priced with the interest rate γ. This allows to compute the funding cash flows $\mathcal{F}$.

Recall that we assume that the cash borrowed from the external funders is fairly priced, and that there is zero recovery for the external funders when the bank defaults.

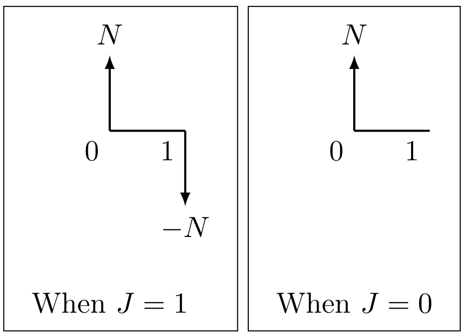
Proposition 2.4.1. *Borrowing money to the external funders comes with interest rate* γ.
In other words, borrowing N *in cash at time* $t = 0$ *from the external funders costs* γN *to the bank.*
The funding cash-flows, when the bank borrows N*, is then*

$$\mathcal{F} = \gamma N - (1 - J)N,$$

and we have

$$\begin{aligned}\mathcal{F}^\circ &= J\gamma N, \\ \mathcal{F}^\bullet &= (1 - J)(1 - \gamma)N.\end{aligned}$$

The realised cash-flows of this transaction (without its price and from the bank's point of view) are, whether the bank has defaulted or not,



Since the pricing is fair with respect to $\mathbb{Q}^\star$, the price p is the expected value of these cash-flows:

$$\begin{aligned}p &= \mathbb{E}^\star(N - JN) \\ &= N\mathbb{E}^\star(1 - J) \\ &= N\mathbb{Q}^\star(J = 0) \\ &= \gamma N.\end{aligned}$$

Hence the funding cash-flows (including the funding price) from the bank to the external funders are

$$\begin{aligned}\mathcal{F} &= -N + \gamma N + JN \\ &= \gamma N - (1-J)N.\end{aligned}$$

We finally obtain

$$\mathcal{F}^\circ = J\mathcal{F} = J\gamma N,$$

and

$$\mathcal{F}^\bullet = -(1-J)\mathcal{F} = (1-J)(N-\gamma N) = (1-J)(1-\gamma)N.$$

2.5 Loss of the bank

In this section, we compute the random variable representing the loss (or profit) the bank (or more precisely its clean desk and CA desk) has to deal with at the end of the deal. Here, we forget the KVA incoming flow at $t=0$. Indeed, KVA will then be calibrated using this loss, to satisfy to regulatory constraints, the shareholders have to put capital at risk in order to cover exceptional losses, and expect to be remunerated for this risk. Their capital at risk is thus computed as an risk measure of the loss, and KVA is seen as a risk premium on this capital at risk.

Proposition 2.5.1. *Let L be the loss (if positive) of the bank at the end of the deal. We have*

$$L = \mathcal{C} + \mathcal{F} - \mathrm{CA}.$$

Since we are here interested in the loss of the bank as a whole, we are not interested in the internal cash-flows. At $t=0$, it is enough to say that the bank pays $(\mathrm{MtM}-\mathrm{CA})^+$ or recieved $(\mathrm{MtM}-\mathrm{CA})^-$ from the client to enter the deal. To do so, the bank borrows $(\mathrm{MtM}-\mathrm{CA})^+$ or lends risk-free $(\mathrm{MtM}-\mathrm{CA})^-$.

At time $t=1$, the bank has a hedging loss $\mathcal{H}$ according to its hedging strategy, pays $\gamma(\mathrm{MtM}-\mathrm{CA})^+$ for its borrowing and gets back $(\mathrm{MtM}-\mathrm{CA})^-$, independently of its default status. Moreover, we have:

- $J = J_1 = 1$: the promised cash-flows $\mathcal{P}$ are exchanged and the bank pays back its funding debt $(\mathrm{MtM}-\mathrm{CA})^+$.
- $J = 1, J_1 = 0$: the bank only pays $\mathcal{P}^-$ and does not recieve $\mathcal{P}^+$ as the client defaulted, and the bank pays back its funding debt $(\mathrm{MtM}-\mathrm{CA})^+$.
- $J = 0, J_1 = 1$: the bank only recieves $\mathcal{P}^+$ and does not pay $\mathcal{P}^-$ nor its funding debt as it has defaulted.

- $J = J_1 = 0$: no further cash-flow is exchanged.

The loss L thus writes

$$\begin{aligned}
L &= \mathcal{H} + \gamma(\mathrm{MtM} - \mathrm{CA})^+ - (\mathrm{MtM} - \mathrm{CA})^- \\
&\quad - JJ_1(\mathcal{P} - (\mathrm{MtM} - \mathrm{CA})^+) + J(1 - J_1)(\mathcal{P}^- + (\mathrm{MtM} - \mathrm{CA})^+) - (1 - J)J_1\mathcal{P}^+ \\
&= \mathcal{H} + \gamma(\mathrm{MtM} - \mathrm{CA})^+ - (\mathrm{MtM} - \mathrm{CA})^- + J(\mathrm{MtM} - \mathrm{CA})^+ \\
&\quad - JJ_1\mathcal{P}^+ + JJ_1\mathcal{P}^- + J\mathcal{P}^- - JJ_1\mathcal{P}^- - J_1\mathcal{P}^+ + JJ_1\mathcal{P}^+ \\
&= \mathcal{H} + \gamma(\mathrm{MtM} - \mathrm{CA})^+ - (\mathrm{MtM} - \mathrm{CA})^- + J(\mathrm{MtM} - \mathrm{CA})^+ + J\mathcal{P}^- - J_1\mathcal{P}^+ \\
&= \mathcal{H} + \gamma(\mathrm{MtM} - \mathrm{CA})^+ - (\mathrm{MtM} - \mathrm{CA})^- + J(\mathrm{MtM} - \mathrm{CA})^+ - \mathcal{P} + \mathcal{C},
\end{aligned}$$

as $J\mathcal{P}^- - J_1\mathcal{P}^+ = -(J_1\mathcal{P}^+ - J\mathcal{P}^-) = -(\mathcal{P} - \mathcal{C}) = -\mathcal{P} + \mathcal{C}$ according to Proposition 2.3.1.

The perfect hedge assumption reads $\mathcal{H} = \mathcal{P} - \mathrm{MtM}$, hence

$$\begin{aligned}
L &= \gamma(\mathrm{MtM} - \mathrm{CA})^+ - (\mathrm{MtM} - \mathrm{CA})^- + J(\mathrm{MtM} - \mathrm{CA})^+ + \mathcal{C} - \mathrm{MtM} \\
&= \gamma(\mathrm{MtM} - \mathrm{CA})^+ - (\mathrm{MtM} - \mathrm{CA})^- + J(\mathrm{MtM} - \mathrm{CA})^+ + \mathcal{C} - (\mathrm{MtM} - \mathrm{CA}) - \mathrm{CA} \\
&= \gamma(\mathrm{MtM} - \mathrm{CA})^+ + J(\mathrm{MtM} - \mathrm{CA})^+ + \mathcal{C} - (\mathrm{MtM} - \mathrm{CA})^+ - \mathrm{CA} \\
&= \gamma(\mathrm{MtM} - \mathrm{CA})^+ - (1 - J)(\mathrm{MtM} - \mathrm{CA})^+ + \mathcal{C} - \mathrm{CA}.
\end{aligned}$$

We thus obtain, using Proposition 2.4.1 with $N = (\mathrm{MtM} - \mathrm{CA})^+$,

$$L = \mathcal{C} + \mathcal{F} - \mathrm{CA},$$

which proves the theorem.

We also have, still by Proposition 2.4.1 with $N = (\mathrm{MtM} - \mathrm{CA})^+$,

$$\begin{aligned}
L^\circ &= J\gamma(\mathrm{MtM} - \mathrm{CA})^+ + \mathcal{C}^\circ - J\mathrm{CA} \\
&= \mathcal{F}^\circ + \mathcal{C}^\circ - J\mathrm{CA}
\end{aligned}$$

and,

$$\begin{aligned}
L^\bullet &= -(1 - J)\gamma(\mathrm{MtM} - \mathrm{CA})^+ + (1 - J)(\mathrm{MtM} - \mathrm{CA})^+ + \mathcal{C}^\bullet + (1 - J)\mathrm{CA} \\
&= (1 - J)(1 - \gamma)(\mathrm{MtM} - \mathrm{CA})^+ + \mathcal{C}^\bullet + (1 - J)\mathrm{CA} \\
&= \mathcal{F}^\bullet + \mathcal{C}^\bullet + (1 - J)\mathrm{CA}.
\end{aligned}$$

Remark 2.5.2. *Note that the pre-default loss L° can be greater than the* CET1*, meaning that negative equity is allowed in our model, without necessarily triggering bank default: this is interpreted as recapitalization. One can easily modify our model to exclude negative equity and recapitalization. This would model the default of the bank as the event $\{L = \mathrm{CET1}\}$, where the loss L would be bounded above by* CET1*. In this approach, the goal is not to model the default of the bank as a solvency issue, as it is more a liquidity one, due to unpaid cash-flow, which can occur even if the bank has capital. The bank default is exogenously calibrated using the CDS curve of the bank.*

2.6 Computing the XVAs and implications

2.6.1 Contra-assets valuation: the equations

We compute in this section the Mark to Market value MtM for the deal and the value of the contra-assets CA, which decomposes as the sum of CVA and FVA, which respectively stand for the Counterparty Value Adjustment and the Funding Value Adjustment.

First, by the Volcker rule, we previously saw that the bank has to be perfectly hedged in order not to do propriatary trading, hence we deduced the relationship $\mathrm{MtM} = \mathbb{E}^\star[\mathcal{P}]$.

In addition, the bank being controlled by the shareholders at time $t = 0$ since it has not defaulted (yet), they compute the prices and rebates MtM, CVA and FVA as to be the fair price for the corresponding cash-flows that impact the shareholders (in particular, these cash-flows may or may not be paid or recieved by the client, but are triggered by the deal and its counterparty and funding implications).

We then show that these definitions allow for the loss L to be centered.

Recall that, in case of default of a party, the position is liquidated. At the default time $1^- = 0$, the clean margin account (which contains CM = MtM) becomes the property of the clean desk, and during the liquidation period, the CVA desk pays the unpaid cash-flows to the clean desk, together with the variations of the mark-to-market of the deal after the default.

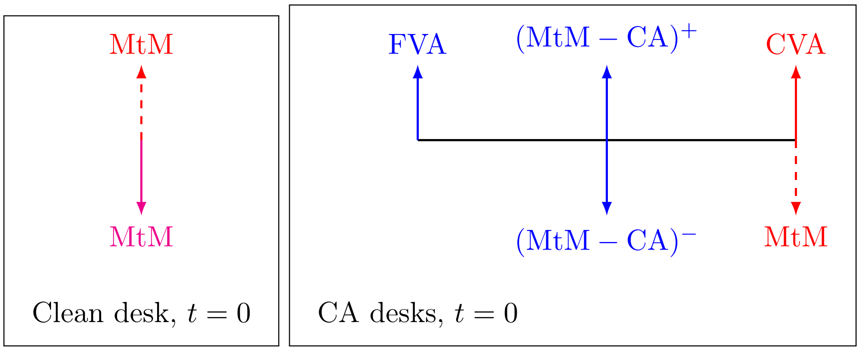
Proposition 2.6.1. *We have*

$$\begin{aligned}
\mathrm{MtM} &= \mathbb{E}^\star\left[\mathcal{P}^\circ + (1-J)\mathrm{MtM}\right] = \mathbb{E}\left[\mathcal{P}\right] = \mathbb{E}\left[\mathcal{P}^\circ\right],\\
\mathrm{CVA} &= \mathbb{E}^\star\left[\mathcal{C}^\circ + (1-J)\mathrm{CVA}\right] = \mathbb{E}\left[\mathcal{C}\right] = \mathbb{E}\left[\mathcal{C}^\circ\right] = \mathbb{E}\left[(1-J_1)\mathcal{P}^+\right],\\
\mathrm{FVA} &= \mathbb{E}^\star\left[\mathcal{F}^\circ + (1-J)\mathrm{FVA}\right] = \mathbb{E}\left[\mathcal{F}\right] = \mathbb{E}\left[\mathcal{F}^\circ\right] = \gamma(\mathrm{MtM} - \mathrm{CVA} - \mathrm{FVA})^+ = \frac{\gamma}{1+\gamma}(\mathrm{MtM} - \mathrm{CVA})^+.
\end{aligned}$$

In particular, the contra-assets valuation is given by

$$\mathrm{CA} = \mathbb{E}^\star\left[\mathcal{C}^\circ + \mathcal{F}^\circ + (1-J)\mathrm{CA}\right].$$

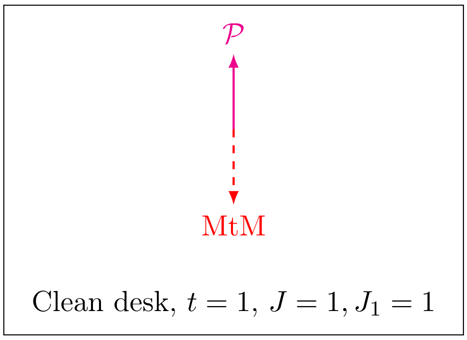
At time $t = 0$, we recall that the client pays CA to the CA desks (CVA to the CVA desk and FVA to the FVA desk), posted in the reserve capital account. Since the reserve capital account is the only funding source, the FVA desk needs to borrow $(\mathrm{MtM} - \mathrm{CA})^+$ from the external funders (with price $\gamma(\mathrm{MtM} - \mathrm{CA})^+$) or lend risk-free $(\mathrm{MtM} - \mathrm{CA})^-$. The CA desks own $\mathrm{CA} + (\mathrm{MtM} + \mathrm{CA})^+ - (\mathrm{MtM} - \mathrm{CA})^- = \mathrm{CA} + \mathrm{MtM} - \mathrm{CA} = \mathrm{MtM}$, posted in the clean margin account and used by the clean desk to pay the client.

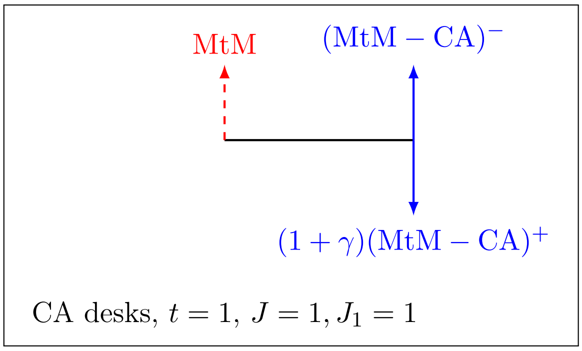


Also recall that the property of the (residual) reserve capital is transferred from the shareholders to the bondholders when the bank defaults.
Last, the laws of *pari-passu* type ensure that the bondholders are hit by the post-default of the bank cash-flows, and not the shareholders.

The shareholders now want to know what is the capital $\mathrm{CM} = \mathrm{MtM}$ to post as collateral, and which represents the clean price, and what is the reserve capital $\mathrm{RC} = \mathrm{CA}$, decomposed into $\mathrm{RC}(C) = \mathrm{CVA}$ and $\mathrm{RC}(F) = \mathrm{FVA}$.
$\mathrm{CM} = \mathrm{MtM}$ is thus the fair price for the shareholders sensitive clean desk cash flows, $\mathrm{RC}(C) = \mathrm{CVA}$ is the fair price for the shareholders sensitive CVA desk cash flows, and $\mathrm{RC}(F) = \mathrm{FVA}$ is the fair price for the shareholders sensitive FVA deks cash flows.
The cash-flows are as follows:

- $J = J_1 = 1$: the bank has not defaulted, the shareholders are thus hit by all the cash-flows. The client also has not defaulted, thus the promised cash flows $\mathcal{P}$ are exchanged between the client and the bank, so the CVA desk has nothing to do. The FVA desk reimburses its debt $(1+\gamma)(\mathrm{MtM}-\mathrm{CA})^+$ or gets back $(\mathrm{MtM}-\mathrm{CA})^-$. The dashed arrow represent the clean margin account, which belongs to the CA desks.

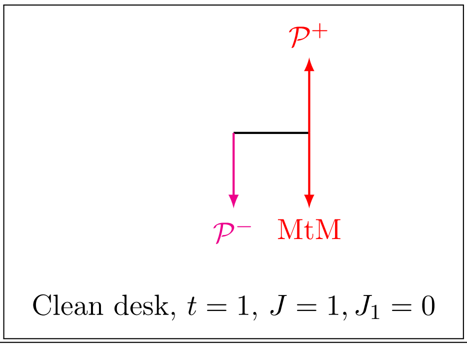


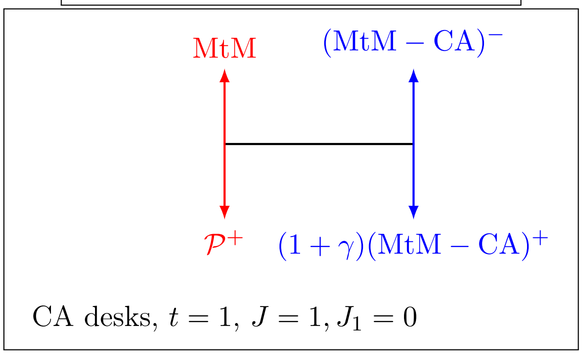


- $J = 1$, $J_1 = 0$: the bank has not defaulted, the shareholders are thus hit by all the cash-flows. The client has defaulted at time $t = 1$, so the clean margin account becomes the property of the clean desk at time $1^- = 0$. Its amount is CM = MtM. At time $t = 1$, the client does not pay $\mathcal{P}^+$ as it has defaulted, the bank still pays $\mathcal{P}^-$. The CVA desk pays $\mathcal{P}^+$ to the clean desk so that it does not see the client's default, and it compensated the MtM fluctuations during the liquidation period (here from $t = 1^- = 0$ to $t = 1$), which goes from MtM at time $t = 1^- = 0$ to 0 at time 0, so it pays $\Delta\text{MtM} = -\text{MtM}$. To summarize, the clean desk recieves:

$$\text{MtM} + \mathcal{P}^+ - \mathcal{P}^- + (-\text{MtM}) = \mathcal{P}.$$

The situation for the FVA desk is as in the case $J = 1 = J_1$.

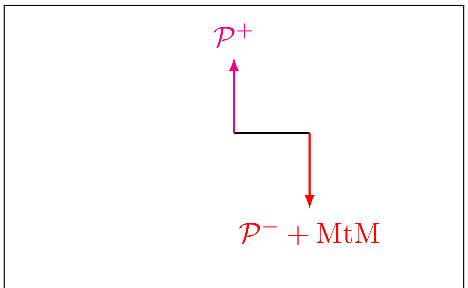




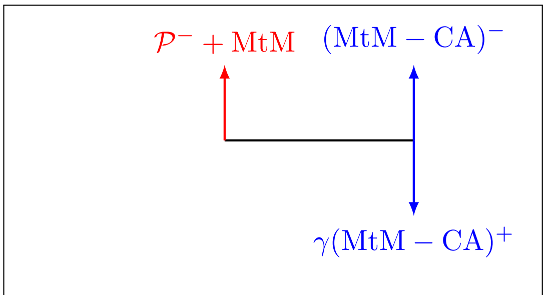
- $J = 0, J_1 = 1$: the bank has defaulted, the shareholders are hit by pre-default cash-flows only. At time $t = 0 = 1^-$, the clean margin account becomes the clean desk property, with value $\mathrm{CM} = \mathrm{MtM}$, and the positions of the CA desks and clean desk are then under the bondholders' authority. At time $t = 1$, the FVA desk do not pay back $(\mathrm{MtM} - \mathrm{CA})^+$ but only the interest $\gamma(\mathrm{MtM} - \mathrm{CA})^+$ as the bank has defaulted. The clean desk recieves $\mathcal{P}^+$ from the client which has not defaulted and pays $\mathcal{P}^-$ to the CVA desk. The CVA pays $-\mathrm{MtM}$ to compensate the post-default MtM fluctuations. As above, the clean desk recieves:

$$\mathrm{MtM} + \mathcal{P}^+ - \mathcal{P}^- + (-\mathrm{MtM}) = \mathcal{P},$$

but here it is important to understand that only the first term is a cash-flow hit by the shareholders, while all the others are hit by the bondholders. The FVA desk only pays $\gamma(\mathrm{MtM} - \mathrm{CA})^+$.

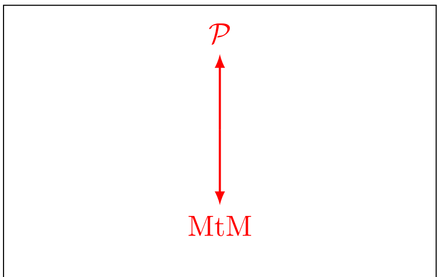


Clean desk, $t = 1$, $J = 0, J_1 = 1$

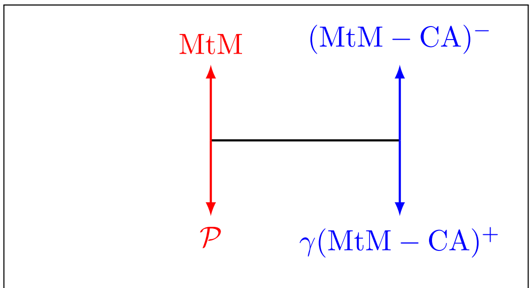


CA desks, $t = 1$, $J = 0, J_1 = 1$

- $J = J_1 = 0$: the situation is the same as before, except that the cash-flow $\mathcal{P}^+$ is paid by the CVA desk and not by the client, as it has also defaulted.



Clean desk, $t = 1$, $J = J_1 = 0$



CA desks, $t = 1$, $J = J_1 = 0$

We now collect the cash-flows for each desk, and we split the cash-flows into the pre-bank default ones and the post-bank default ones, as the pre-bank default ones are shareholder sensitive, while the post-bank default ones are bondholder sensitive.

- Clean desk:

$$\mathcal{P} - \mathrm{MtM} = J(\mathcal{P} - \mathrm{MtM}) + (1 - J)(\mathrm{MtM} - \mathrm{MtM} + \mathcal{P} - \mathrm{MtM}),$$

where, for the second term, the pre bank default cash-flows are MtM − MtM (paiment of MtM to the client compensated by the clean margin account, which becomes the clean desk's property), and the post bank default cash-flows are $\mathcal{P}-$MtM. Thus the shareholders are hit by the cash-flows. The pre-default cash-flows are thus

$$\begin{aligned}
&J(\mathcal{P} - \mathrm{MtM}) + (1 - J)(\mathrm{MtM} - \mathrm{MtM})\\
&= J\mathcal{P} + (1 - J)\mathrm{MtM} - \mathrm{MtM}\\
&= \mathcal{P}^{\circ} + (1 - J)\mathrm{MtM} - \mathrm{MtM},
\end{aligned}$$

and the post-default ones are $(1 - J)(\mathcal{P} - \mathrm{MtM})$.

- CVA desk:

$$\begin{aligned}
&\mathrm{CVA} - \mathrm{MtM} + JJ_1\mathrm{MtM} + J(1 - J_1)(\mathrm{MtM} - \mathcal{P}^+)\\
&\qquad + (1 - J)J_1(\mathcal{P}^- + \mathrm{MtM}) + (1 - J)(1 - J_1)(\mathrm{MtM} - \mathcal{P})\\
&= \mathrm{CVA} - J(1 - J_1)\mathcal{P}^+ + (1 - J)(J_1\mathcal{P}^- - (1 - J_1)\mathcal{P})\\
&= \mathrm{CVA} - J(1 - J_1)\mathcal{P}^+ + (1 - J)(J_1\mathcal{P}^- - \mathcal{P} + J_1\mathcal{P}^+ - J_1\mathcal{P}^-)\\
&= \mathrm{CVA} - J(1 - J_1)\mathcal{P}^+ + (1 - J)(-\mathrm{CVA} + \mathrm{CVA} - \mathcal{P} + J_1\mathcal{P}^+).
\end{aligned}$$

Here, the cash-flows $(1 - J)(-\mathrm{CVA} + \mathrm{CVA})$ are added to account for the fact that the amount of the (CVA part of the) reserve capital account goes from the shareholders to the bondholders. The pre-default cash-flows are thus

$$\begin{aligned}
&\mathrm{CVA} - J(1 - J_1)\mathcal{P}^+ - (1 - J)\mathrm{CVA}\\
&= \mathrm{CVA} - \mathcal{C}^{\circ} - (1 - J)\mathrm{CVA},
\end{aligned}$$

recall 2.3.1, and the post-default ones are

$$\begin{aligned}
&(1 - J)(\mathrm{CVA} - \mathcal{P} + J_1\mathcal{P}^+)\\
&= (1 - J)\mathrm{CVA} + (1 - J)(J_1\mathcal{P}^+ - \mathcal{P}^+ + \mathcal{P}^-)\\
&= (1 - J)\mathrm{CVA} + (1 - J)(\mathcal{P}^- - (1 - J_1)\mathcal{P}^+)\\
&= (1 - J)\mathrm{CVA} + \mathcal{C}^{\bullet},
\end{aligned}$$

recall again 2.3.1.

- FVA desk:

$$\begin{aligned}
&\mathrm{FVA} + (\mathrm{MtM} - \mathrm{CA})^+ - (\mathrm{MtM} - \mathrm{CA})^- + J((\mathrm{MtM} - \mathrm{CA})^- - (1 + \gamma)(\mathrm{MtM} - \mathrm{CA})^+)\\
&\quad + (1 - J)((\mathrm{MtM} - \mathrm{CA})^- - \gamma(\mathrm{MtM} - \mathrm{CA})^+)\\
&= \mathrm{FVA} + J(\mathrm{MtM} - \mathrm{CA})^+ + (1 - J)(\mathrm{MtM} - \mathrm{CA})^+ - J(\mathrm{MtM} - \mathrm{CA})^+\\
&\quad - J\gamma(\mathrm{MtM} - \mathrm{CA})^+ - (1 - J)\gamma(\mathrm{MtM} - \mathrm{CA})^+\\
&= \mathrm{FVA} - J\gamma(\mathrm{MtM} - \mathrm{CA})^+ + (1 - J)(1 - \gamma)(\mathrm{MtM} - \mathrm{CA})^+\\
&= \mathrm{FVA} - J\gamma(\mathrm{MtM} - \mathrm{CA})^+ + (1 - J)(-\mathrm{FVA} + \mathrm{FVA} + (1 - \gamma)(\mathrm{MtM} - \mathrm{CA})^+)
\end{aligned}$$

Here, the cash-flows $(1-J)(-\mathrm{FVA}+\mathrm{FVA})$ are added to account for the fact that the amount of the (FVA part of the) reserve capital account goes from the shareholders to the bondholders. The pre-default cash-flows are thus

$$\begin{aligned}&\mathrm{FVA} - J\gamma(\mathrm{MtM} - \mathrm{CA})^+ - (1-J)\mathrm{FVA}\\&= \mathrm{FVA} - \mathcal{F}^\circ - (1-J)\mathrm{FVA},\end{aligned}$$

recall 2.4.1, and the post-default ones are

$$\begin{aligned}&(1-J)(\mathrm{FVA} + (1-\gamma)(\mathrm{MtM} - \mathrm{CA})^+)\\&= (1-J)\mathrm{FVA} + \mathcal{F}^\bullet,\end{aligned}$$

recall again 2.4.1.

As the shareholders take the investment decisions at $t = 0$, they compute MtM and the various add-ons as to make the cash-flows for the three desks centered

- The MtM is computed as the $\mathbb{Q}^\star$-expectation of the shareholders gains generated by the clean desk, hence.

$$\begin{aligned}0 &= \mathbb{E}^\star\left[\mathcal{P}^\circ + (1-J)\mathrm{MtM} - \mathrm{MtM}\right], \text{ i.e.}\\\mathrm{MtM} &= \mathbb{E}^\star\left[\mathcal{P}^\circ + (1-J)\mathrm{MtM}\right].\end{aligned}$$

- The CVA is computed as the $\mathbb{Q}^\star$-expected shareholder loss generated by the CVA desk, hence:

$$\begin{aligned}0 &= \mathbb{E}^\star\left[\mathrm{CVA} - \mathcal{C}^\circ - (1-J)\mathrm{CVA}\right], \text{ i.e.}\\\mathrm{CVA} &= \mathbb{E}^\star\left[\mathcal{C}^\circ + (1-J)\mathrm{CVA}\right].\end{aligned}$$

- The FVA is computed as the $\mathbb{Q}^\star$-expected shareholder loss generated by the FVA desk, hence:

$$\begin{aligned}0 &= \mathbb{E}^\star\left[\mathrm{FVA} - \mathcal{F}^\circ - (1-J)\mathrm{FVA}\right], \text{ i.e.}\\\mathrm{FVA} &= \mathbb{E}^\star\left[\mathcal{F}^\circ + (1-J)\mathrm{FVA}\right].\end{aligned}$$

Note that the previous identities for MtM, CVA and FVA are *equations*, hence need to be solved.

2.6.2 Solution to the MtM, CVA and FVA equations

We define the *bank survival measure* $\mathbb{Q}$ as, for each event $A \in \mathcal{A}$,

$$\mathbb{Q}(A) := \mathbb{Q}^\star(A \mid J = 1) = \frac{\mathbb{Q}^\star(A \cap \{J = 1\})}{\mathbb{Q}^\star(J = 1)} = \frac{\mathbb{Q}^\star(A \cap \{J = 1\})}{1-\gamma}.$$

In particular, if $\mathbb{E}^\star$ (resp. $\mathbb{E}$) denotes the $\mathbb{Q}^\star$-expectation (resp. $\mathbb{Q}$-expectation, we have, for any random variable $\mathcal{Y}$,

$$\mathbb{E}(\mathcal{Y}) = \frac{\mathbb{E}^\star(\mathcal{Y}1_{J=1})}{1-\gamma} = \frac{\mathbb{E}^\star(J\mathcal{Y})}{1-\gamma}.$$

For a random variable $\mathcal{Y}$, we set $\mathcal{Y}^\circ := J\mathcal{Y}$ and $Y^\bullet := -(1-J)\mathcal{Y}$ so that $Y = \mathcal{Y}^\circ - \mathcal{Y}^\bullet$. The financial interpretion is as follows: the shareholders are only concerned by cash-flows occuring before the bank default, so, given a random loss for example, $\mathcal{Y}^\circ$ is the shareholder sensitive loss. The post-default cash-flows are destinated to the bondholders, so $\mathcal{Y}^\bullet$ is the gain for the bondholders.

Using this notation, we have

$$\mathbb{E}(\mathcal{Y}) = \frac{\mathbb{E}^\star(\mathcal{Y}^\circ)}{1-\gamma}.$$

We have the following easy lemma.

Lemma 2.6.2. *Let $y \in \mathbb{R}$ and $\mathcal{Y}$ be a random variable. Then $\mathbb{E}(\mathcal{Y}) = \mathbb{E}(\mathcal{Y}^\circ)$ and*

$$y = \mathbb{E}^\star(\mathcal{Y}^\circ + (1-J)y) \Longleftrightarrow y = \mathbb{E}(\mathcal{Y}).$$

Proof. Since $(\mathcal{Y}^\circ)^\circ = JJY = JY = \mathcal{Y}^\circ$, we have

$$\mathbb{E}(\mathcal{Y}^\circ) = \frac{\mathbb{E}^\star((\mathcal{Y}^\circ)^\circ)}{1-\gamma} = \frac{\mathbb{E}^\star(\mathcal{Y}^\circ)}{1-\gamma} = \mathbb{E}(\mathcal{Y}).$$

In addition, we have

$$\begin{aligned} y = \mathbb{E}^\star(\mathcal{Y}^\circ + (1-J)y) &\Longleftrightarrow y = \mathbb{E}^\star(\mathcal{Y}^\circ) + \mathbb{E}^\star(1-J)y \\ &\Longleftrightarrow y = \mathbb{E}^\star(\mathcal{Y}^\circ) + \mathbb{Q}^\star(J=0)y \\ &\Longleftrightarrow (1-\gamma)y = \mathbb{E}^\star(\mathcal{Y}^\circ) \\ &\Longleftrightarrow y = \frac{\mathbb{E}^\star(\mathcal{Y}^\circ)}{1-\gamma} \\ &\Longleftrightarrow y = \mathbb{E}(\mathcal{Y}). \end{aligned}$$

□

Using Lemma 2.6.2 and Propositions 2.3.1 and 2.4.1, we obtain

$$\begin{aligned} \text{MtM} &= \mathbb{E}\left[\mathcal{P}\right] = \mathbb{E}\left[\mathcal{P}^\circ\right], \\ \text{CVA} &= \mathbb{E}\left[\mathcal{C}\right] = \mathbb{E}\left[\mathcal{C}^\circ\right] = \mathbb{E}\left[J(1-J_1)\mathcal{P}^+\right] = \mathbb{E}\left[(1-J_1)\mathcal{P}^+\right], \\ \text{FVA} &= \mathbb{E}\left[\mathcal{F}\right] = \mathbb{E}\left[\mathcal{F}^\circ\right] = \mathbb{E}\left[J\gamma(\text{MtM} - \text{CA})^+\right] \\ &= \mathbb{E}\left[\gamma(\text{MtM} - \text{CVA} - \text{FVA})^+\right] = \gamma(\text{MtM} - \text{CVA} - \text{FVA})^+. \end{aligned}$$

The last identity is still a semi-linear equation for FVA, which we can solve

$$\text{FVA} = \frac{\gamma}{1+\gamma}(\text{MtM} - \text{CVA})^+.$$

2.6.3 Contra-liabilities value

In addition, we can also compute the contra-liabilities value CL, which is bondholders value after the bank's default.
Since contra-liabilities is only value to the bondholders and not to the shareholders, who make the investment decisions, they are not willing to pay for these future gains, as they are not allowed to monetize these future gains beforehand due to *pari-passu* type laws. Note that the bondholders won't make profits by the clean desk as the market risks are perfectly hedged, according to the Volcker rule.

We now collect the post-bank default cash flows, which are the contra-liabilities of the bank, and which value CL is decomposed into the Debt Value Adjustment DVA, corresponding to the CVA desk expected gains, and the Funding Debt Adjustment FDA, corresponding to FVA post-default profits.

- CVA desk:

$$\mathrm{DVA} = \mathbb{E}^{\star}\left[(1-J)\mathrm{CVA} + \mathcal{C}^{\bullet}\right].$$

- FVA desk:

$$\mathrm{FDA} = \mathbb{E}^{\star}\left[\mathcal{F}^{\bullet} + (1-J)\mathrm{FVA}\right].$$

2.6.4 Consequences

We give a few consequences of the previous results. Namely, we show that the shareholders loss is centered, that the benefits of the bondholders from the risky funding is exactly equal to the cost of risky funding from the shareholders point of view, and that the firm valuation of counterparty risk is, as expected, the cost for the shareholders (CA) minus the profit for the bondholders (CL), and also the cost coming from the client's default risk (CVA) minus the gain from the bank own default (DVA).

Proposition 2.6.3. *We have*

$$\begin{aligned}
&\mathbb{E}^{\star}\left[L^{\circ}\right] = \mathbb{E}\left[L^{\circ}\right] = \mathbb{E}\left[L\right] = 0,\\
&\mathrm{FDA} = \mathrm{FVA},\\
&\mathrm{FV} = \mathrm{CVA} - \mathrm{DVA} = \mathrm{CA} - \mathrm{CL}
\end{aligned}$$

We observe that these values for MtM, CVA and FVA allow for the shareholder loss to be centered with respect to $\mathbb{Q}^{\star}$ and to $\mathbb{Q}$. Indeed, we have

$$\begin{aligned}
\mathrm{CVA} &= \mathbb{E}^{\star}\left[\mathcal{C}^{\circ} + (1-J)\mathrm{CVA}\right] = \mathrm{CVA} + \mathbb{E}^{\star}\left[\mathcal{C}^{\circ} - J\mathrm{CVA}\right],\\
\mathrm{FVA} &= \mathbb{E}^{\star}\left[\mathcal{F}^{\circ} + (1-J)\mathrm{FVA}\right] = \mathrm{FVA} + \mathbb{E}^{\star}\left[\mathcal{F}^{\circ} - J\mathrm{FVA}\right],
\end{aligned}$$

hence

$$\mathbb{E}^{\star}\left[\mathcal{C}^{\circ} - J\mathrm{CVA}\right] = \mathbb{E}^{\star}\left[\mathcal{F}^{\circ} - J\mathrm{FVA}\right] = 0.$$

Now, using Proposition 2.5.1, we have

$$\begin{aligned}\mathbb{E}^\star\left[L^\circ\right] &= \mathbb{E}^\star\left[\mathcal{C}^\circ + \mathcal{F}^\circ - J\mathrm{CA}\right] \\ &= \mathbb{E}^\star\left[\mathcal{C}^\circ - J\mathrm{CVA}\right] + \mathbb{E}^\star\left[\mathcal{F}^\circ - J\mathrm{FVA}\right] \\ &= 0.\end{aligned}$$

Hence, by definition and by Lemma 2.6.2, we obtain

$$\mathbb{E}\left[L^\circ\right] = \mathbb{E}\left[L\right] = \frac{\mathbb{E}^\star\left[L^\circ\right]}{1-\gamma} = 0.$$

Moreover, since $\mathbb{E}^\star\left[\mathcal{F}\right] = 0$, we obtain

$$\mathrm{FVA} - \mathrm{FDA} = \mathbb{E}^\star\left[\mathcal{F}^\circ + (1-J)\mathrm{FVA}\right] - \mathbb{E}^\star\left[\mathcal{F}^\bullet + (1-J)\mathrm{FVA}\right] = \mathbb{E}^\star\left[\mathcal{F}\right] = 0,$$

hence FVA = FDA.

The firm valuation FV of counterparty risk is thus:

$$\begin{aligned}\mathrm{FV} &= \mathbb{E}^\star\left[\mathcal{C} + \mathcal{F}\right] \\ &= \mathbb{E}^\star\left[\mathcal{C}\right] \\ &= \mathbb{E}^\star\left[\mathcal{C}^\circ\right] - \mathbb{E}^\star\left[\mathcal{C}^\bullet\right] \\ &= \mathbb{E}^\star\left[J\mathrm{CVA} - \mathcal{C}^\bullet\right]\end{aligned}$$

Using the definition for DVA, we obtain

$$\begin{aligned}\mathrm{DVA} &= \mathbb{E}^\star\left[\mathcal{C}^\bullet + (1-J)\mathrm{CVA}\right] \\ &= \mathrm{CVA} + \mathbb{E}^\star\left[\mathcal{C}^\bullet - J\mathrm{CVA}\right],\end{aligned}$$

which gives

$$\mathrm{CVA} - \mathrm{DVA} = \mathbb{E}^\star\left[J\mathrm{CVA} - \mathcal{C}^\bullet\right] = \mathrm{FV},$$

and thus, using that FVA = FDA,

$$\begin{aligned}\mathrm{FV} &= \mathrm{CVA} - \mathrm{DVA} \\ &= \mathrm{CVA} + \mathrm{FVA} - \mathrm{FDA} - \mathrm{DVA} \\ &= \mathrm{CA} - \mathrm{CL}.\end{aligned}$$

2.7 Capital Value Adjustment KVA

In the previous sections, we defined MtM, CA = CVA + DVA so that the pre-bank default cash-flows for each desk (clean desk, CVA desk and FVA desk) are fairly priced, and we saw that the shareholders' loss is centered, i.e. that $\mathbb{E}^\star[L^\circ] = 0$. However, L° is not zero: the shareholders still have to face losses, which might be exceptionally important. In order to be sure that the bank is able to pay its debts, the regulators want the shareholders to put some capital at risk (Economic Capital EC). Since this capital is immobilized by the shareholders to absorb losses, they want expect the bank to pay a risk premium at a hurdle rate h on their capital at risk. This risk, assumed by the shareholders, exists because of the deal: if the deal was not contracted, the loss would exactly be equal to 0 and no capital at risk would be necessary. To provide this risk premium to the shareholders, the bank thus includes it in the pricing as a further value adjustment, called capital value adjustment and noted KVA, on top of the counterparty and funding value adjustments. This KVA amount paid by the client is put in the risk margin account, which, as stated in the financial assumptions, is also loss-absorbing, meaning that it is part of the capital at risk and can be used by the bank as immobilized capital by the shareholders to absorb losses.

The Economic Capital EC, defined as the amount that regulators want to see immobilized for loss absorption, is computed as a risk measure on the loss, assuming that the bank will not default (meaning that it is computed under the surviva probability $\mathbb{Q}$). More precisely, EC is defined as the expected shortfall at level 97.5% of the shareholders' loss L° under $\mathbb{Q}$:

$$\mathrm{EC} = \mathbb{ES}_{0.975}\left[L^\circ\right] = \frac{1}{1-0.975}\int_{0.975}^{1} \mathrm{VaR}_\alpha(L^\circ)d\alpha,$$

where, for $0 < \alpha < 1$,

$$\mathrm{VaR}_\alpha(L^\circ) = \inf\left\{x \in \mathbb{R} \mid \mathbb{Q}\left(L^\circ \leq x\right) \geq \alpha\right\},$$

meaning that, the loss is smaller than $x = \mathrm{VaR}_\alpha(L^\circ)$ with probability α (close to 1).

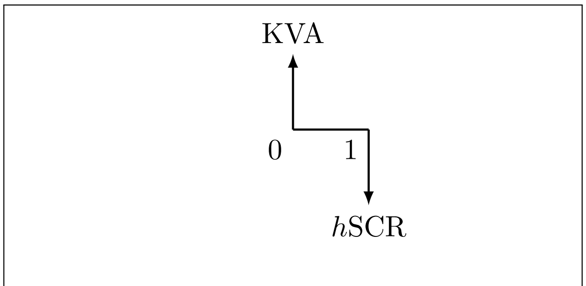
Since the regulators want to see EC immobilized for loss absorption, and since the capital value adjustment KVA (to be determined) is also loss-absorbing, the capital at risk that need to be provided by shareholders (shareholders capital at risk SCR) is given by:

$$\mathrm{SCR} = \max(\mathrm{EC}, \mathrm{KVA}) - \mathrm{KVA} = (\mathrm{EC} - \mathrm{KVA})^+.$$

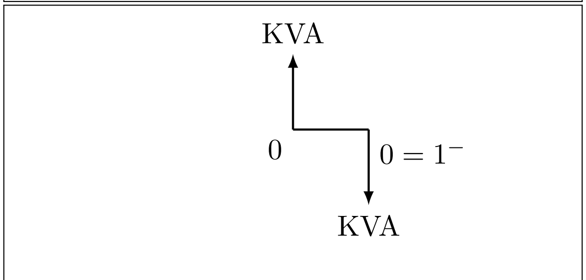
We also define the full amount of capital at risk CR by

$$\mathrm{CR} = \mathrm{SCR} + \mathrm{KVA} = \max(\mathrm{EC}, \mathrm{KVA}).$$

If the bank is not in default, the management of the bank, responsible for the capital at risk, uses the risk margin KVA to reward the shareholders on their capital at risk SCR with rate h. Otherwise, if the bank defaults, then the risk margin KVA becomes the bondholders' property. The shareholders sensitive cash flows of the management desk are thus:



KVA desk (shareholders sensitive), $J = 1$



KVA desk (shareholders sensitive), $J = 0$

Hence, KVA is defined as

$$\begin{aligned}\mathrm{KVA} &= \mathbb{E}^{\star}\left[Jh\mathrm{SCR} + (1-J)\mathrm{KVA}\right] \\ &= \mathbb{E}^{\star}\left[h\mathrm{SCR}^{\circ} + (1-J)\mathrm{KVA}\right] \\ &= \mathrm{RM} = \mathrm{RM}^{sh} + \mathrm{RM}^{bh},\end{aligned}$$

with $\mathrm{RM}^{sh} = \mathbb{E}^{\star}\left[h\mathrm{SCR}^{\circ}\right]$ and $\mathrm{RM}^{bh} = \mathbb{E}^{\star}\left[(1-J)\mathrm{KVA}\right]$ are the respective expected costs for the bank, respectively to the shareholders and the bondholders. This expected cost is covered by the KVA payment at $t = 0$ by the client.

Using again Lemma 2.6.2 and the fact that $\mathrm{SCR} = (\mathrm{EC} - \mathrm{KVA})^{+}$ is a deterministic constant, we can solve the equation for KVA:

$$\mathrm{KVA} = \mathbb{E}\left[h\mathrm{SCR}\right] = h(\mathrm{EC} - \mathrm{KVA})^{+}.$$

As in the case of the FVA, this semilinear equation admits the solution

$$\mathrm{KVA} = h\mathrm{EC}^{+} = \frac{h}{1+h}\mathrm{EC} = \frac{h}{1+h}\mathbb{ES}_{0.975}\left[L^{\circ}\right].$$

2.7.1 A connection to risk aversion

We are going to see that this risk margin, and more precisely the hurdle rate h at which is remunerated the shareholder capital at risk, is closely linked to their risk aversion.

We know that the shareholder trading loss L° is centered by the definition of MtM, CVA and FVA. Assume that the shareholders want to compute the indifference price of the deal.

If U is a utility function with $U(0) = 0$, the indifference price is such that

$$\mathbb{E}^\star \left[U(J(\mathrm{RM} - L))\right] = \mathbb{E}^\star \left[U(0)\right] = 0,$$

meaning that the shareholders are indifferent between entering the deal and recieving back $\mathrm{RM} - L$ if the bank has not defaulted, and doing nothing.

We have, as $U(0) = 0$,

$$\mathbb{E}^\star \left[U(J(\mathrm{RM} - L))\right] = \mathbb{E}^\star \left[JU(\mathrm{RM} - L)\right] = (1-\gamma)\frac{\mathbb{E}^\star \left[JU(\mathrm{RM} - L)\right]}{1-\gamma} = (1-\gamma)\mathbb{E}\left[U(\mathrm{RM} - L)\right].$$

Thus, the indifference price is such that

$$\mathbb{E}\left[U(\mathrm{RM} - L)\right] = 0.$$

Taking $U(-\ell) = \frac{1-e^{-\rho\ell}}{\rho}$ with $\rho > 0$ a risk aversion parameter, we obtain:

$$0 = \mathbb{E}\left[\frac{1 - e^{-\rho(\mathrm{RM}-L)}}{\gamma}\right] = 1 - e^{-\rho\mathrm{RM}}\mathbb{E}\left[e^{\rho L}\right],$$

thus

$$\mathrm{RM} = \frac{\ln\left(\mathbb{E}\left[e^{\rho L}\right]\right)}{\rho}.$$

In view of the previous computations for $\mathrm{KVA} = \mathrm{RM}$, we obtain

$$\frac{h}{1+h} = \frac{\ln\left(\mathbb{E}\left[e^{\rho L}\right]\right)}{\rho\mathrm{EC}}.$$

Taking h and ρ close to 0, one obtains

$$h \simeq \frac{\mathrm{VaR}(L^\circ)}{2\mathrm{EC}}\rho.$$

2.8 Funds transfer price and wealth transfer analysis

2.8.1 Funds transfer price FTP

We conclude by summarizing the XVA rebates that were computed, and we analyse these prices under a wealth transfer point of view.

The sum of all XVA rebates is called funds transfer prices and denoted FTP. It is thus equal to the sum of the counterparty value adjustment, the funding valuation adjustment and the capital value adjustment:

$$\mathrm{FTP} = \mathrm{CVA} + \mathrm{FVA} + \mathrm{KVA} = \mathrm{CA} + \mathrm{KVA},$$

which is the sum of the expected cost for the CVA and FVA desks, responsible for the contra-assets, and the capital value adjustment used to remunerate the shareholders.

Using the fact that FVA = FDA, by inserting DVA, we obtain

$$\mathrm{FTP} = (\mathrm{CVA} - \mathrm{DVA}) + (\mathrm{DVA} + \mathrm{FDA}) + \mathrm{KVA},$$

which is the sum of the firm valuation of counterparty risk, the expected bondholders gains from the contra-liabilities, which are a wealth transfer from the shareholders to the bondholders, and again the shareholders risk premium.

2.8.2 Wealth transfer analysis

All the previous results we obtained are under the natural hypothesis that the bank cannot hedge its own jump-to-default using a further deal. Assume, for the sake of the argument, that it is possible: the bank recieves at time $t = 0$ the payment CL and delivers at time $t = 1$ the cash-flow $L^\bullet$.

In this context, the price of the deal would now be $\mathrm{MtM} - \mathrm{FV} = \mathrm{MtM} - \mathrm{CA} + \mathrm{CL}$.

At time $t = 0$, the bank still needs to borrow only $(\mathrm{MtM} - \mathrm{CA})^+$, and use CL from the new deal to have $\mathrm{MtM} - \mathrm{CA} + \mathrm{CL}$ to pay the client, so the flows are still vanishing at time $t = 0$.
At time $t = 1$, the only difference is that the loss is increased by a further $L^\bullet$ due to the new deal. The new loss is then

$$L + L^\bullet = L^\circ,$$

which is also equal to

$$L + L^\bullet = \mathcal{C} + \mathcal{F} - \mathrm{CA} + L^\bullet = \mathcal{C} + \mathcal{F} - (\mathrm{CA} - \mathrm{CL}) + (L^\bullet - \mathrm{CL}) = \mathcal{C} + \mathcal{F} - \mathrm{FV} + (L^\bullet - \mathrm{CL}).$$

In that hypothetical situation, the shareholders are still indifferent to the deal in counterparty and funding as nothing has changed from their point of view. The bondholders are now zero recovery, as their gain due to the bank default $L^\bullet$ has now to be paid due to the additional deal. The client is better off by the amount CL. Furthermore, the bank would still charge the same KVA add-on for the expected shareholders loss L°.

Last, assume that the bank could also enter an other deal at time $t = 0$, where they recieve L°. This deal costs nothing at time $t = 0$ as $\mathbb{E}^\star[L^\circ] = 0$ in this context, and the total loss for the three deals (the original deal, the deal with payment $L^\bullet$ and this deal with payoff L°) is now $L - L^\circ + L^\bullet = 0$, so here the KVA would vanish as the shareholders loss vanishes. As a result, the FTP rebate would here be $\mathrm{FV} = \mathrm{CVA} - \mathrm{DVA}$.

Note that these conclusions are in line with what was announced in Section 1.3.6, when discussing links with the Modigliani-Miller theorem of 1958.

Chapter 3

Continuous time model

As in the static case, we consider a dealer bank having contracting deals with clients. We assume that the bank is in *run-off*, meaning that it will not enter in new deals anymore.

3.1 Probabilistic setup

We consider a probabilistic setup $(\Omega, \mathcal{A}, \mathbb{G}, \mathbb{Q})$ where the sigma-algebra $\mathcal{A}$ encompasses all the available information, in particular the defaults of the bank and of each of its clients.

The probability measure $\mathbb{Q}$ will be used both for pricing and XVA computations, and for risk measures computations, as in the static case.

Here, $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ is a filtration satisfying to the usual conditions. Every process will be $\mathbb{G}$-adapted, and every time will be a $\mathbb{G}$-stopping time.

We denote by T the last maturity of all contracts. We model the bank's default as a totally inaccessible $\mathbb{G}$-stopping time τ. We define $J = 1_{[0,\tau[}$ the survival indicator process of the bank.

We consider the same notations as in the static case, which we recall here:

- $\mathcal{P}$ are the *cumulated promised* cash-flows recieved by the clean desk from the clients and the CVA desk,
- $\mathcal{C}$ are the *cumulated counterparty* cash-flows paid by the CVA desk to the clean desk,
- $\mathcal{F}$ are the *cumulated risky funding* cash-flows paid by the FVA desk to the external funders,
- $\mathcal{H}$ are the *cumulated hedging* cash-flows from the bank to the hedging market.

We assume that $\mathcal{P}_0 = \mathcal{C}_0 = \mathcal{F}_0$.

Given a process $\mathcal{Y}$, we set

$$\mathcal{Y}_t^{\tau^-} = \mathcal{Y}_{\tau^- \wedge t} = \mathcal{Y}_t 1_{[0,\tau)}(t) + \mathcal{Y}_{\tau^-}(1 - 1_{[0,\tau)}(t)),$$

i.e.

$$\mathcal{Y}^{\tau^-} = J\mathcal{Y} + (1-J)\mathcal{Y}_{\tau^-}.$$

Moreover, we define

$${}^{\tau^-}\mathcal{Y} = \mathcal{Y} - \mathcal{Y}^{\tau^-} = (1-J)(\mathcal{Y} - \mathcal{Y}_{\tau^-}),$$

We thus have ${}^{\tau^-}\mathcal{Y}_t = 0$ if $t < \tau$ and ${}^{\tau^-}\mathcal{Y}_t = \mathcal{Y}_t - \mathcal{Y}_{\tau^-}$ if $t \geq \tau$, and $\mathcal{Y} = \mathcal{Y}^{\tau^-} + {}^{\tau^-}\mathcal{Y}$.

It is natural to assume that the processes $\mathcal{C}^{\tau^-}$ and $\mathcal{F}^{\tau^-}$ are non-decreasing: since these processes are cumulative cash flows, before the bank's default, the clients defaults and the CVA/FVA desks have to pay the cash-flows related to these defauls.

3.2 Desks cash-flows

In this section, building on the results we observed in the static case, we define the cash-flows for each desk of the bank.

The cash-flows for the clean desk are:

- $\mathcal{P}$ from the client or the CVA desk thanks to the clean margin account,
- $\mathrm{MtM} - \mathrm{MtM}_0$ is the fluctuation of the marked-to-model "mark-to-market" account of the bank.
- The hedging loss Π.

The profit process for the clean desk is thus

$$\begin{aligned}
\mathcal{P} + \mathrm{MtM} - \mathrm{MtM}_0 - \Pi &= J(\mathcal{P} + \mathrm{MtM} - \mathrm{MtM}_0 - \Pi) + (1-J)(\mathcal{P} + \mathrm{MtM} - \mathrm{MtM}_0 - \Pi) \\
&= J(\mathcal{P} + \mathrm{MtM} - \mathrm{MtM}_0 - \Pi) \\
&\quad + (1-J)(\mathcal{P} - \mathcal{P}_{\tau^-} + \mathrm{MtM} - \mathrm{MtM}_{\tau^-} + \Pi - \Pi_{\tau^-} + \\
&\qquad \mathcal{P}_{\tau^-} + \mathrm{MtM}_{\tau^-} + \Pi_{\tau^-} - \mathrm{MtM}_0) \\
&= (\mathcal{P} - \mathcal{P}_{\cdot\wedge\tau^-} + \mathrm{MtM} - \mathrm{MtM}_{\cdot\wedge\tau^-}) + (\mathcal{P}_{\cdot\wedge\tau^-} + \mathrm{MtM}_{\cdot\wedge\tau^-} - \mathrm{MtM}_0) \\
&= ({}^{\tau^-}\mathcal{P} + {}^{\tau^-}\mathrm{MtM} + {}^{\tau^-}\Pi) + (\mathcal{P}^{\tau^-} + \mathrm{MtM}^{\tau^-} - \mathrm{MtM}_0 - \Pi^{\tau^-})
\end{aligned}$$

The second parenthesis $\mathcal{P}^{\tau^-} + \mathrm{MtM}^{\tau^-} - \mathrm{MtM}_0 - \Pi^{\tau^-}$ are the pre-default cumulative cash-flows, which are destinated to the shareholders, consisting in the cumulative promised cash-flows $\mathcal{P}$ before default, obtained from the clients or the CVA desk, together with the pre-default mark-to-market fluctuations and the clean hedging loss. The first parenthesis consists in the post-default cumulative promised cash-flows ${}^{\tau^-}\mathcal{P}_t = \mathcal{P}_t - \mathcal{P}_{\tau^-}$ for $t \geq \tau$, together with post-default mark-to-market fluctuations and post-default clean hedging loss.

Similarly, the cumulative cash-flows for the CVA desk is

- $\mathcal{C}$ to the clean desk due to defaults,
- $\mathrm{CVA} - \mathrm{CVA}_0$ the fluctuations of the marked-to-model contra-asset CVA account, which reflects the expected future cost due to counterparty risk,
- Φ is the hedging gain of the CVA desk.

The loss process of the CVA desk is

$$\mathcal{C} + \mathrm{CVA} - \mathrm{CVA}_0 - \Phi = ({}^{\tau^-}\mathcal{C} + {}^{\tau^-}\mathrm{CVA} - {}^{\tau^-}\Phi) + (\mathcal{C}^{\tau^-} + \mathrm{CVA}^{\tau^-} - \mathrm{CVA}_0 - \Phi^{\tau^-}),$$

where the distinction between pre and post-default cash-flows is similar.

Last, the FVA desk cash-flows are

- $\mathcal{F}$ to the external funders,
- $\mathrm{FVA} - \mathrm{FVA}_0$ the fluctuations of the marked-to-model contra-asset FVA account, which reflects the expected future cost due to the funding implications of counterparty risk,
- Ψ is the hedging gain of the FVA desk.

The loss process for the FVA desk is given by

$$\mathcal{F} + \mathrm{FVA} - \mathrm{FVA}_0 - \Psi = ({}^{\tau^-}\mathcal{F} + {}^{\tau^-}\mathrm{FVA} - {}^{\tau^-}\Psi) + (\mathcal{F}^{\tau^-} + \mathrm{FVA}^{\tau^-} - \mathrm{FVA}_0 - \Psi^{\tau^-}),$$

with still the same interpretation.

Of course, the processes MtM, CVA and FVA are still to be defined.

We consider the process $\mathcal{H}$ which represents the hedging cash flows, inclusive of the price to set up the strategy, which thus decomposes into

$$\mathcal{H} = \Pi - \Phi - \Psi.$$

We assume that the processes $\Pi^{\tau^-}, {}^{\tau^-}\Pi, \Phi^{\tau^-}, {}^{\tau^-}\Phi, \Psi^{\tau^-}, {}^{\tau^-}\Psi$ are all $(\mathbb{G}, \mathbb{Q})$-martingales. In particular, the processes $\mathcal{H}^{\tau^-} = \Pi^{\tau^-} + \Phi^{\tau^-} + \Psi^{\tau^-}$, ${}^{\tau^-}\mathcal{H} = {}^{\tau^-}\Pi + {}^{\tau^-}\Phi + {}^{\tau^-}\Psi$, $\Pi = \Pi^{\tau^-} + {}^{\tau^-}\Pi$, $\Phi = \Phi^{\tau^-} + {}^{\tau^-}\Phi$ and $\Psi = \Psi^{\tau^-} + {}^{\tau^-}\Psi$ and $\mathcal{H} = \Pi + \Phi + \Psi$ are all $(\mathbb{G}, \mathbb{Q})$-martingales.

The loss process of the bank is thus:

$$\begin{aligned}\mathcal{L} = &-(\mathcal{P} + \mathrm{MtM} - \mathrm{MtM}_0 - \Pi) + \\ &(\mathcal{C} + \mathrm{CVA} - \mathrm{CVA}_0 - \Phi) + \\ &(\mathcal{F} + \mathrm{FVA} - \mathrm{FVA}_0 - \Psi) \\ = &\,\mathcal{C} + \mathcal{F} + \mathrm{CA} - \mathrm{CA}_0 - (\mathcal{P} + \mathrm{MtM} - \mathrm{MtM}_0) + \mathcal{H}.\end{aligned}$$

Example 3.2.1. *The* reference hedging case *is when the CA desks are not hedged* ($\Phi = \Psi = 0$). *The bank hedging loss* $\mathcal{H}$ *thus coincides with the heding loss* Π *of the clean desks, which are perfectly hedged, meaning that*

$$\Pi = \mathcal{P} + \mathrm{MtM} - \mathrm{MtM}_0.$$

In that case, the loss process becomes

$$\mathcal{L} = \mathcal{C} + \mathcal{F} + \mathrm{CA} - \mathrm{CA}_0.$$

Example 3.2.2. *Assume that the bank enters at* $t = 0$ *into a long call option with payoff* $(S_T - K)^+$ *(where* S *is the price process for the underlying Black & Scholes asset), and is delta-hedged. Then the promised cash-flows is*

$$\mathcal{P} = 1_{[T,\infty)} (S_T - K)^+ .$$

Let P *the Black & Sholes proce process of this call option, for* $t \leq T$*:*

$$\begin{aligned} P_t = P(t, S_t) &= \inf \left\{ y \geq 0 \mid \exists \zeta, y + \int_0^{\cdot} \zeta_t dS_t = \left(S_T^{t,S_t} - K \right)^+ \right\} \\ &= \mathbb{E}_t \left[(S_T - K)^+ \right], \end{aligned}$$

and $P_t = 0$ *for* $t \geq T$. *It is well-known that the optimal process* ζ *is, in this situation, the Black & Scholes delta of the option, given by:*

$$\zeta_t = \partial_x P(t, S_t)$$

and, for all $t \in [0, \infty)$*, we have the usual dynamic programming equation*

$$P_0 + \int_0^t \zeta_s dS_s = (S_T - K)^+ 1_{[T,\infty)}(t) + P_t.$$

the clean desk gains are (inclusive of the hedge):

$$\mathcal{P} + P - P_0 - \int_0^{\cdot} \zeta_t dS_t,$$

which is thus identically equal to 0 *as* ζ *is the Black & Scholes delta of the option.*

3.3 The MtM, CVA and FVA processes

We now define the processes MtM, CVA and FVA. As in the static case, we are going to define these processes as to make the shareholder sensitive losses centered $(\mathbb{G}, \mathbb{Q})$-martingales for each of the three desks, and hence the full loss process stopped before bank default $\mathcal{L}^{\tau^-}$.

3.3.1 Value

If $\mathcal{Y}$ is an optional integrable process stopped at T, we define its *value process* the process Z such that

$$Z_t = \mathbb{E}_t\left[\mathcal{Y}_T - \mathcal{Y}_t\right], \quad t \le T.$$

In particular, we have $Z_t = 0$ for $t \ge T$ and $Z + \mathcal{Y}$ is a $(\mathbb{G}, \mathbb{Q})$-martingale.

We observe that Z is solution to

$$\begin{aligned} &Z_t = 0, \quad t \ge T, \\ &dZ_t = -d\mathcal{Y}_t + d\mu_t, \quad t \le T, \end{aligned}$$

for some $(\mathbb{G}, \mathbb{Q})$-martingale μ on $[0, T]$.

3.3.2 Shareholder value

If $\mathcal{Y}$ is an optional integrable process, we define its *shareholder value process* as any process Z such that, for all $t \ge 0$,

$$\begin{aligned} &1_{\{t<\tau\}} Z_t = \mathbb{E}_t\left[\mathcal{Y}_{\tau^-} - \mathcal{Y}_t + Z_{\tau^-}\right] 1_{\{t<\tau\}}, \\ &Z_t 1_{\{T<\tau\}} 1_{\{t \ge T\}} = 0. \end{aligned}$$

First, observe that we say "any process" as the shareholder value of $\mathcal{Y}$ is not uniquely defined: it is defined on $\{t < \tau\}$, and on $\{t \ge T\} \cap \{T < \tau\}$. Hence the shareholder value is not defined on $\{t \ge \tau\} \cap \{\tau \le T\}$. Indeed, this is because, from a shareholders point of view, what happens at and after the default of the bank is irrelevant. We will later define the MtM and XVA processes as shareholder value for some processes, and we will specify the values on $\{t \ge \tau\} \cap \{\tau \le T\}$ appropriately.

Thus the definition is only interesting before τ: if $\tau \le T$, no definition is given at and after τ. If $T \le \tau$, then the value is 0 at and after T (and consequently at and after τ also). We can restrict our attention to Z^{τ^-} on $[0, T \wedge \tau]$: if $\tau \le T$, $Z^{\tau^-} = Z$ on $[0, \tau \wedge T) = [0, \tau)$ and $Z^{\tau^-}_\tau = Z^{\tau^-}_{\tau^-}$, while if $T < \tau$, then $Z^{\tau^-} = Z$ on $[0, \tau \wedge T] = [0, T]$ with $Z^{\tau^-}_T = Z_T = 0$.

Now, observe that if $Z_t 1_{\{T<\tau\}} 1_{\{t \ge T\}} = 0$, then Z^{τ^-} solves

$$Z^{\tau^-}_t = \mathbb{E}_t\left[\mathcal{Y}^{\tau^-}_{\tau\wedge T} - \mathcal{Y}^{\tau^-}_t + 1_{\{\tau \le T\}} Z^{\tau^-}_{\tau \wedge T}\right], \quad t \le \tau \wedge T,$$

as this equation is equivalent to the first one. By extension, we will call Z^{τ^-} on $[0, \tau \wedge T]$ the shareholder value of $\mathcal{Y}$.

Hence this implies that $(Z + \mathcal{Y})^{\tau^-}$ is a $(\mathbb{G}, \mathbb{Q})$-martingale stopped before τ.

We thus observe that Z^{τ^-} on $[0, T \wedge \tau]$ is a solution to

$$\begin{aligned} &Z^{\tau^-}_T 1_{\{T<\tau\}} 0, \\ &dZ^{\tau^-}_t = -d\left(\mathcal{Y}\right)^{\tau^-}_t + d\nu_t, \quad t \le T \wedge \tau. \end{aligned}$$

where ν is a $(\mathbb{G}, \mathbb{Q})$-martingale on $[0, \tau \wedge T]$ stopped before τ.

3.3.3 The MtM, CVA, FVA processes

We define MtM (resp. CVA, FVA) as the shareholder value process of $\mathcal{P}$ (resp. $\mathcal{C}$, $\mathcal{F}$), meaning that $\mathrm{MtM}_t = \mathrm{CVA}_t = \mathrm{FVA}_t = 0$ if $t \geq T$ and $\tau > T$, and for $t < \tau$,

$$\begin{aligned}\mathrm{MtM}_t &= \mathbb{E}_t\left[\mathcal{P}_{\tau^-} - \mathcal{P}_t + \mathrm{MtM}_{\tau^-}\right],\\ \mathrm{CVA}_t &= \mathbb{E}_t\left[\mathcal{C}_{\tau^-} - \mathcal{C}_t + \mathrm{CVA}_{\tau^-}\right],\\ \mathrm{FVA}_t &= \mathbb{E}_t\left[\mathcal{F}_{\tau^-} - \mathcal{F}_t + \mathrm{FVA}_{\tau^-}\right].\end{aligned}$$

We assume that solutions to these equations exist and unique. We will later one provide a setup and conditions on the cash-flows processes, such that these equations are well-posed.

Remember that the shareholders are only hit by pre-bank default cash-flows. Under these definitions, $(\mathrm{MtM} + \mathcal{P})^{\tau^-}$, $(\mathrm{CVA} + \mathcal{C})^{\tau^-}$ and $(\mathrm{FVA} + \mathcal{F})^{\tau^-}$ are martingales, and remember that we also assume that Π^{τ^-}, Φ^{τ^-} and Ψ^{τ^-} are also assumed to be martingales. Thus we readily observe that the trading losses of each desk, stopped before τ, is a martingale. Hence the loss process of the bank shareholders $\mathcal{L}^{\tau^-}$ is a martingale, which motivates the above definitions.

From the previous discussion, we notice that the processes MtM, CVA and FVA are unconstrained on $\{t \geq T\} \cap \{\tau \leq T\}$.

We set $\mathrm{CVA}_t = \mathrm{FVA}_t = 0$ on this set. In particular, we observe that, by the definition of a shareholder value, we also have $\mathrm{CVA}_t = \mathrm{FVA}_t = 0$ on $\{t \geq T\} \cap \{T < \tau\}$. Hence $\mathrm{CVA}_t = \mathrm{FVA}_t = 0$ on $\{t \geq T \wedge \tau\}$. In particular, we have

$$\begin{aligned}{}^{\tau^-}\mathrm{CVA} &= 1_{[\tau,\infty)}\left(\mathrm{CVA} - \mathrm{CVA}_{\tau^-}\right) = -1_{[\tau,\infty)}\mathrm{CVA}_{\tau^-}, \text{ and}\\ {}^{\tau^-}\mathrm{FVA} &= 1_{[\tau,\infty)}\left(\mathrm{FVA} - \mathrm{FVA}_{\tau^-}\right) = -1_{[\tau,\infty)}\mathrm{FVA}_{\tau^-}.\end{aligned}$$

This explains why we are only interested in CVA^{τ^-} and FVA^{τ^-}, which satisfy to

$$\begin{aligned}\mathrm{CVA}^{\tau^-}_t &= \mathrm{CVA}_{\tau^-}1_{\{\tau\leq T\}}, \quad t \geq \tau \wedge T\\ \mathrm{CVA}^{\tau^-}_t &= \mathbb{E}_t\left[\mathcal{C}^{\tau^-}_{\tau\wedge T} - \mathcal{C}^{\tau^-}_t + 1_{\{\tau\leq T\}}\mathrm{CVA}^{\tau^-}_{\tau\wedge T}\right], \quad t \leq \tau \wedge T,\end{aligned}$$

and similarly

$$\begin{aligned}\mathrm{FVA}^{\tau^-}_t &= \mathrm{FVA}_{\tau^-}1_{\{\tau\leq T\}}, \quad t \geq \tau \wedge T\\ \mathrm{FVA}^{\tau^-}_t &= \mathbb{E}_t\left[\mathcal{F}^{\tau^-}_{\tau\wedge T} - \mathcal{F}^{\tau^-}_t + 1_{\{\tau\leq T\}}\mathrm{FVA}^{\tau^-}_{\tau\wedge T}\right], \quad t \leq \tau \wedge T.\end{aligned}$$

Regarding the MtM process, we assume in addition that MtM is the value process of $\mathcal{P}$, i.e.

$$\begin{aligned}\mathrm{MtM}_t &= 0, \quad t \geq T,\\ \mathrm{MtM}_t &= \mathbb{E}_t\left[\mathcal{P}_T - \mathcal{P}_t\right], \quad t \leq T.\end{aligned}$$

Indeed, remember that we assumed that Π^{τ^-} and ${}^{\tau^-}\Pi$ are $(\mathbb{G},\mathbb{Q})$-martingale, hence $\Pi = \Pi^{\tau^-} + {}^{\tau^-}\Pi$ is also one. The reference hedging case is thus compatible with our previous assumption only under the assumption that MtM is the value process of $\mathcal{P}$.

3.4 The KVA process

As in the static case, we observe that the CVA and FVA processes are defined as to make the shareholder loss process a martingale. However, since contra-assets are difficult to be replicated, not to mention contra-liabilities, whose replication is forbidden, losses are not equal to zero and exceptional shareholder losses can still occur. Some capital then needs to be set at risk by shareholders. They therefore deserve a further risk premium, which is the KVA add-on.

The Economic Capital is the amount that the regulator wants to see on an economic basis. Since KVA is loss-absorbing in our setting, the actual Capital at Risk is $\mathrm{CR} = \max(\mathrm{EC}, \mathrm{KVA})$, and the shareholders capital at risk is thus

$$\mathrm{SCR} = \mathrm{CR} - \mathrm{KVA} = \max(\mathrm{EC}, \mathrm{KVA}) - \mathrm{KVA} = (\mathrm{EC} - \mathrm{KVA})^+.$$

The KVA process is defined as the shareholder value process for the process $\int_0^{\cdot} h\mathrm{SCR}_s ds$, i.e.

$$\mathrm{KVA}_t = \mathbb{E}_t\left[\int_t^{\tau^-} h\left(\mathrm{EC}_s - \mathrm{KVA}_s\right)^+ ds + \mathrm{KVA}_{\tau^-}\right], \quad t < \tau,$$

and $\mathrm{KVA}_t 1_{t \geq \tau \wedge T} = 0$. We assume that this equation admits a unique solution. We will see later a setup where we can prove this.

We observe that KVA^{τ^-} is a supermartingale, with drift coefficient $-h\mathrm{SCR} = -h(\mathrm{EC} - \mathrm{KVA})^+$. The previous equation can be written in the following backward differential form:

$$\begin{aligned} &1_{T<\tau}\mathrm{KVA}_T^{\tau^-} = 0, \\ &d\mathrm{KVA}_t^{\tau^-} = -h\mathrm{SCR}_t dt + d\nu_t, \quad \text{for } t \leq \tau \wedge T, \end{aligned}$$

where ν is a martingale.

The financial interpretation is the continuous counterpart as the one given in the static case: KVA_t is the amount to the bank has to have on its risk margin account in order to be able to dynamically deliver a hurdle rate h on the shareholder capital at risk. If no default, this account should zero at maturity T: all the money has been used to remunerate the shareholders. If $\mathrm{KVA}_T 1_{T<\tau} < 0$, this means that the risk margin is not sufficient to remunerate the shareholders, while if this quantity is positive, it means that the add-on paid by the clients is unnecessarily high in order to remunerate the shareholders.

The shareholders dividends are thus $-(\mathcal{L}^{\tau^-} + \mathrm{KVA}^{\tau^-} - \mathrm{KVA}_0)$, which is a submartingale stopped before τ, with drift coefficient $h\mathrm{SCR}$.

3.5 Solving the XVA equations

3.5.1 Main assumptions

We assume the following:

- There exists a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ such that
  - $\mathcal{F}_t \subset \mathcal{G}_t$ for all $t \geq 0$.
  - $\mathbb{F}$ satisfies to the usual conditions.
  - τ is not a $\mathbb{F}$-stopping time.
  - Given a $\mathbb{G}$-predictable process Y on $[0, \tau \wedge T]$, there exists a $\mathbb{F}$-predictable process Y' on $[0, T]$, such that $Y 1_{[0,\tau]} = Y' 1_{[0,\tau]}$. Y' is called the $\mathbb{F}$-(predictable) reduction of Y.
  - Given a $\mathbb{G}$-optional process Y on $[0, \tau \wedge T)$, there exists a $\mathbb{F}$-predictable process Y' on $[0, T]$, such that $Y 1_{[0,\tau)} = Y' 1_{[0,\tau)}$. Y' is called the $\mathbb{F}$-(optional) reduction of Y.
  - $\mathbb{Q}(\tau > T \mid \mathcal{F}_T) > 0$.
- There exists a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ such that
  - $\mathbb{P}$ is equivalent to $\mathbb{Q}_{|\mathcal{F}_T}$.
  - Given a $(\mathbb{F}, \mathbb{P})$-local martingale M on $[0, T]$, M^{τ^-} is a $(\mathbb{G}, \mathbb{Q})$-local martingale on $[0, \tau \wedge T]$.
  - Conversely, if M is a $(\mathbb{G}, \mathbb{Q})$-local martingale on $[0, \tau \wedge T]$ without jump at τ (i.e. $M_\tau - M_{\tau^-} = 0$), M' is a $(\mathbb{F}, \mathbb{P})$-local martingale on $[0, T]$.

Then τ is called an "invariance time" and $\mathbb{P}$ an "invariance probability measure".

Under the hypothesis that $\mathbb{Q}(\tau > T \mid \mathcal{F}_T) > 0$, then the optional reductions are uniquely defined on $[0, T]$.

Moreover, the last assumptions on $\mathbb{F}$ imply that for each $\mathbb{G}$-stopping time θ, there exists a unique $\mathbb{F}$-stopping time θ' such that $\theta \wedge \tau = \theta' \wedge \tau$. Taking $\theta = \tau$, we observe that $\tau \wedge \tau = \tau = (+\infty) \wedge \tau$ and that $+\infty$ is indeed a $\mathbb{F}$-stopping time, so that $\tau' = +\infty$.

A classical setup, called "progressive enlargment", is where $\mathbb{F}$ satisfies to the usual conditions, τ is a random time which is not a $\mathbb{F}$-stopping time, and the filtration $\mathbb{G}$ is defined with $\mathcal{G}_t = \mathcal{F}_t \vee \{\tau \leq t\}$ for all $t \geq 0$. In this setting, one can show that the predictable and optional reductions exist. Under the additional assumption that $\mathbb{Q}(\tau > T \mid \mathcal{F}_T) > 0$, one can show that uniqueness holds.

As in the static case, there is a link with the bank survival probability measure associated with $\mathbb{Q}$. More precisely:

Theorem 3.5.1. *Assume in addition that τ has a $(\mathbb{G},\mathbb{Q})$-intensity process $\gamma = \gamma 1_{[0,\tau]}$ such that $e^{\int_0^\tau \gamma_s ds}$ is $\mathbb{Q}$-integrable.*

Then there exists a unique *invariance probability measure $\mathbb{P}$, which is equal to the restriction to $\mathcal{F}_T$ of $\mathbb{Q}(\cdot \mid \tau > T)$, the bank survival probability measure associated to $\mathbb{Q}$.*

As mentionned in the static case, the case where $\mathbb{P} \neq \mathbb{Q}$ represents hard wrong way risk situations between the bank's default and its positions and/or between the bank's default and a client's one.

In the following, we denote by $\mathbb{E}'[\cdot]$ the $(\mathbb{F},\mathbb{P})$-expectation.

Last, let us define $\mathbb{S}_2$ as the space of $\mathbb{G}$-adapted càdlàg processes Y on $[0, \tau \wedge T]$ without jump at τ and such that

$$\|Y\|^2_{\mathbb{S}^2} := \mathbb{E}\left[Y_0^2 + \int_0^T J_s e^{\int_0^s \gamma_u du} d(Y^\star)^2_s\right] < +\infty,$$

where $Y^\star_s = \sup_{s\in[0,t]} |Y_s|$. It is in fact possible to prove that

$$\|Y\|^2_{\mathbb{S}^2} = \mathbb{E}'\left[\sup_{t\in[0,T]} (Y'_t)^2\right],$$

which explains the notation $\mathbb{S}^2$. Moreover, for $Y \in \mathbb{S}^2$, we have

$$\mathbb{E}\left[\sup_{t\in[0,T\wedge\tau]} Y_t^2\right] \leq \|Y\|^2_{\mathbb{S}^2} < +\infty.$$

We also define $\mathbb{S}'_2$ as the space of $\mathbb{F}$-adapted progressive processes Y' on $[0,T]$ such that

$$\|Y'\|^2_{\mathbb{S}'_2} := \mathbb{E}'\left[\sup_{t\in[0,T]} (Y'_t)^2\right] < +\infty.$$

In particular, the $\mathbb{F}$-reduction is an isometry from $\mathbb{S}_2$ onto $\mathbb{S}'_2$ with stopping before τ as reciprocal opertor.

3.5.2 Reduced equations

We recall the equations for MtM^{τ^-}, CVA^{τ^-}, FVA^{τ^-} and KVA^{τ^-}. We have $\mathrm{MtM}_T 1_{T<\tau} = \mathrm{CVA}_T 1_{T<\tau} = \mathrm{FVA}_T 1_{T<\tau} = \mathrm{KVA}_T 1_{T<\tau} = 0$ and, for $t \leq \tau \wedge T$,

$$\begin{aligned}
\mathrm{MtM}^{\tau^-}_t &= \mathbb{E}_t\left[\mathcal{P}^{\tau^-}_{\tau\wedge T} - \mathcal{P}^{\tau^-}_t + 1_{\tau\leq T}\mathrm{MtM}^{\tau^-}_\tau\right],\\
\mathrm{CVA}^{\tau^-}_t &= \mathbb{E}_t\left[\mathcal{C}^{\tau^-}_{\tau\wedge T} - \mathcal{C}^{\tau^-}_t + 1_{\tau\leq T}\mathrm{CVA}^{\tau^-}_\tau\right],\\
\mathrm{FVA}^{\tau^-}_t &= \mathbb{E}_t\left[\mathcal{F}^{\tau^-}_{\tau\wedge T} - \mathcal{F}^{\tau^-}_t + 1_{\tau\leq T}\mathrm{FVA}^{\tau^-}_\tau\right],\\
\mathrm{KVA}^{\tau^-}_t &= \mathbb{E}_t\left[\int_t^{\tau\wedge T} h\left(\mathrm{EC}_s - \mathrm{KVA}^{\tau^-}_s\right)^+ ds + 1_{\tau\leq T}\mathrm{KVA}^{\tau^-}_\tau\right].
\end{aligned}$$

Observe that the KVA equation is (*a priori*) more involved than the three others as the KVA process is itself appearing in a component of the cash-flows process.

We thus consider a more general case: if $\mathcal{Y}$ is an optional integrable process and if $j : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ is $\mathcal{P}(\mathbb{F}) \times \mathcal{B}(\mathbb{R})$-measurable, we consider the shareholder value associated to $\mathcal{Y}$ and j as a process Z such that $Z_t 1_{\{t \geq T\}} 1_{\{T<\tau\}} = 0$ and, for $t \leq \tau \wedge T$,

$$Z_t^{\tau^-} = \mathbb{E}_t \left[\mathcal{Y}_{\tau\wedge T}^{\tau^-} - \mathcal{Y}_t^{\tau^-} + \int_t^{\tau\wedge T} j_s \left(Z_s^{\tau^-} \right) ds + 1_{\{\tau \leq T\}} Z_\tau^{\tau^-} \right].$$

Of course when $j = 0$, we get back the previously introduced shareholder value associated to $\mathcal{Y}$.

As before, we compute, for $t \leq \tau \wedge T$,

$$Z_t^{\tau^-} = \mathbb{E}_t \left[\mathcal{Y}_{\tau\wedge T}^{\tau^-} - \mathcal{Y}_t^{\tau^-} + \int_0^{\tau\wedge T} j_s \left(Z_s^{\tau^-} \right) ds - \int_0^t j_s \left(Z_s^{\tau^-} \right) ds + 1_{\{\tau \leq T\}} Z_\tau^{\tau^-} \right],$$

so

$$Z_t^{\tau^-} + \mathcal{Y}_t^{\tau^-} + \int_0^t j_s \left(Z_s^{\tau^-} \right) ds = \mathbb{E}_t \left[\mathcal{Y}_{\tau\wedge T}^{\tau^-} + \int_0^{\tau\wedge T} j_s \left(Z_s^{\tau^-} \right) ds + 1_{\{\tau \leq T\}} Z_\tau^{\tau^-} \right],$$

and $Z^{\tau^-} + \mathcal{Y}^{\tau^-} + \int_0^{\cdot} j_s \left(Z_s^{\tau^-} \right) ds$ is a $(\mathbb{G}, \mathbb{Q})$-martingale on $[0, \tau \wedge T]$.

Thus Z^{τ^-} is solution to:

$$\begin{aligned}
&Z_T^{\tau^-} 1_{\{T<\tau\}} = 0, \\
&dZ_t^{\tau^-} = -j_t \left(Z_t^{\tau^-} \right) dt - d\mathcal{Y}_t^{\tau^-} + d\nu_t, \quad t \leq \tau \wedge T,
\end{aligned}$$

for some $(\mathbb{G}, \mathbb{Q})$-martingale ν on $[0, \tau \wedge T]$.

By a $\mathbb{S}_2$ solution to the previous equation, or a $\mathbb{S}_2$ shareholder value process for $\mathcal{Y}$ and j, it is meant a solution such that $\nu \in \mathbb{S}_2$. Equivalently, this amounts to $Z^{\tau^-} + \mathcal{Y}^{\tau^-} + \int_0^{\cdot} j_s \left(Z_s^{\tau^-} \right) ds \in \mathbb{S}_2$.

We now introduce the reduced value associated to $\mathcal{Y}$ and j. Recall that, by assumtion, if $\mathcal{Y}$ is a $\mathbb{Q}$-predictable process, there exists a unique $\mathbb{F}$-predictable process $\mathcal{Y}'$ (called the $\mathbb{F}$-reduction of $\mathcal{Y}$) that coincides with $\mathcal{Y}$ before τ, i.e. $\mathcal{Y}^{\tau^-} = \mathcal{Y}'^{\tau^-}$.

The *reduced value* of $\mathcal{Y}'$ and j is the process Z' satisfying to

$$\begin{aligned}
&Z'_t = 0, \quad t \geq T, \\
&Z'_t = \mathbb{E}'_t \left[\mathcal{Y}'_T - \mathcal{Y}'_t + \int_t^T j_s(Z'_s) ds \right], \quad t \leq T.
\end{aligned}$$

Using that $Z'_T = 0$, we observe that, for $t \leq T$,

$$Z'_t = \mathbb{E}'_t \left[\int_t^T j_s(Z'_s) ds + \mathcal{Y}'_T - \mathcal{Y}'_t + Z'_T \right]$$
$$= \mathbb{E}'_t \left[\int_0^T j_s(Z'_s) ds - \int_0^t j_s(Z'_s) ds + \mathcal{Y}'_T - \mathcal{Y}'_t + Z'_T \right],$$

so

$$Z'_t + \mathcal{Y}'_t + \int_0^t j_s(Z'_s) ds = \mathbb{E}'_t \left[\int_0^T j_s(Z'_s) ds + \mathcal{Y}'_T + Z'_T \right],$$

and $Z' + \mathcal{Y}' + \int_0^{\cdot} j_s(Z'_s) ds$ is a $(\mathbb{F}, \mathbb{P})$-martingale on $[0, T]$.

Thus Z' solves

$$\begin{aligned} &Z'_T = 0, \\ &dZ'_t = -j_t(Z'_t) dt - d\mathcal{Y}'_t + d\mu_t, \quad t \leq T, \end{aligned}$$

where μ is a $(\mathbb{F}, \mathbb{P})$-martingale on $[0, T]$.

By a $\mathbb{S}'_2$ solution to the previous equation, or a $\mathbb{S}'_2$ reduced value process for $\mathcal{Y}$ and j, it is meant a solution such that $\mu \in \mathbb{S}'_2$. Equivalently, this amounts to $Z' + \mathcal{Y}' + \int_0^{\cdot} j_s \left(Z'_s \right) ds \in \mathbb{S}'_2$.

Theorem 3.5.2. *Let $\mathcal{Y}$ be an optional integrable process and j a $\mathcal{P}(\Omega) \times \mathcal{B}(\mathbb{R})$-measurable map.*

Assume that Z^{τ^-} is a $\mathbb{S}_2$ shareholder value process for $\mathcal{Y}$ and j. Then its reduction is a $\mathbb{S}'_2$ reduced value for $\mathcal{Y}'$ and j.
Conversely, if Z' is a $\mathbb{S}'_2$ reducedvalue process for $\mathcal{Y}'$ and j, then $(Z')^{\tau^-}$ is a $\mathbb{S}_2$ shareholder value process for $\mathcal{Y}$ and j.

In particular, the reductions of MtM, CVA, FVA *and* KVA *are solution, on $[0, T]$, to*

$$\begin{aligned} \mathrm{MtM}'_t &= \mathbb{E}'_t \left[\mathcal{P}'_T - \mathcal{P}'_t \right], \\ \mathrm{CVA}'_t &= \mathbb{E}'_t \left[\mathcal{C}'_T - \mathcal{C}'_t \right], \\ \mathrm{FVA}'_t &= \mathbb{E}'_t \left[\mathcal{F}'_T - \mathcal{F}'_t \right], \\ \mathrm{KVA}'_t &= \mathbb{E}'_t \left[\int_t^T h \left(\mathrm{EC}'_s - \mathrm{KVA}'_s \right)^+ ds \right], \end{aligned}$$

with $\mathrm{MtM}'_t 1_{t \geq T} = \mathrm{CVA}'_t 1_{t \geq T} = \mathrm{FVA}'_t 1_{t \geq T} = \mathrm{KVA}'_t 1_{t \geq T} = 0$.

Proof. Let Z^{τ^-} on $[0, \tau \wedge T]$ a $\mathbb{S}_2$ shareholder value for $\mathcal{Y}$ and j. It thus satisfies, for some $(\mathbb{G}, \mathbb{Q})$-martingale on $[0, T \wedge \tau]$ $\nu \in \mathbb{S}_2$,

$$\begin{aligned} &Z_T^{\tau^-} 1_{\{T < \tau\}} = 0, \\ &dZ_t^{\tau^-} = -j_t \left(Z_t^{\tau^-} \right) dt - d\mathcal{Y}_t^{\tau^-} + d\nu_t, \quad t \leq T \wedge \tau. \end{aligned}$$

We thus have, as Z and its reduction coincide before τ, recalling that $\tau' = +\infty$,

$$\begin{aligned}
0 &= \mathbb{E}\left[Z_T^{\tau^-} 1_{\{T<\tau\}} \mid \mathcal{F}_T\right] \\
&= \mathbb{E}\left[(Z')_T^{\tau^-} 1_{\{T<\tau'\}} \mid \mathcal{F}_T\right] \\
&= \mathbb{E}\left[Z'_{\tau^-\wedge T} 1_{\{T<\tau\}} \mid \mathcal{F}_T\right] \\
&= \mathbb{E}\left[Z'_T 1_{\{T<\tau\}} \mid \mathcal{F}_T\right] \\
&= Z'_T \mathbb{Q}\left(T < \tau \mid \mathcal{F}_T\right).
\end{aligned}$$

This gives $Z'_T = 0$ as $\mathbb{Q}(T < \tau \mid \mathcal{F}_T) > 0$ by assumption.

We set $\mu = \nu'$, which is in $\mathbb{S}'_2$ as the $\mathbb{F}$-reduction is an isometry from $\mathbb{S}_2$ to $\mathbb{S}'_2$. Moreover, by assumption, μ is a $(\mathbb{F}, \mathbb{P})$-martingale on $[0, T]$ as ν is a $(\mathbb{G}, \mathbb{Q})$-martingale on $[0, T \wedge \tau]$ without jump at τ, as it is equal to $Z^{\tau^-} + \mathcal{Y}^{\tau^-} + \int_0^\cdot j_s \left(Z_s^{\tau^-}\right) ds$.

By reduction, on $[0, T \wedge \tau]$, we find that $dZ'_t = -j_t(Z'_t)dt - d\mathcal{Y}'_t + d\nu_t$. Since the two processes are equal on $[0, T \wedge \tau]$ and the $\mathbb{F}$-reductions are unique, we obtain the equality on $[0, T]$. Hence Z' is a $\mathbb{S}'_2$ reduced value for $\mathcal{Y}$ and j.

Conversely, assume that Z' is a $\mathbb{S}'_2$ reduced value process for $\mathcal{Y}$ and j. It thus satisfies, for some $(\mathbb{F}, \mathbb{P})$-martingale on $[0, T]$ $\mu \in \mathbb{S}'_2$,

$$\begin{aligned}
&Z'_T = 0, \\
&dZ'_t = -j_t(Z'_t)dt - d\mathcal{Y}'_t + d\mu_t, \quad t \leq T.
\end{aligned}$$

Then, as $(Z')^{\tau^-} = Z^{\tau^-}$, we obtain $Z_T^{\tau^-} 1_{\{T<\tau\}} = (Z')_T^{\tau^-} 1_{\{T<\tau\}} = Z'_{\tau^-\wedge T} 1_{\{T<\tau\}} = Z'_T 1_{\{T<\tau\}} = 0$.

We set $\nu = \mu^{\tau^-}$ which is in $\mathbb{S}_2$ as stopping before τ is an isometry from $\mathbb{S}'_2$ to $\mathbb{S}_2$, and which is, by assumption, a $(\mathbb{G}, \mathbb{Q})$-martingale on $[0, T \wedge \tau]$ as μ is a $(\mathbb{F}, \mathbb{P})$-martingale on $[0, T]$.

Stopping the second equation before τ, we obtain,

$$dZ_t^{\tau^-} = -j_t\left(Z_t^{\tau^-}\right) dt - d\mathcal{Y}_t^{\tau^-} - d\nu_t, \quad t \leq T \wedge \tau,$$

which shows that Z^{τ^-} is a shareholder value for $\mathcal{Y}$ and j. □

Remark 3.5.3. *Since* MtM *is the value for* $\mathcal{P}$, *the equation* $\mathrm{MtM}'_t = \mathbb{E}'_t\left[\mathcal{P}'_T - \mathcal{P}'_t\right]$ *for* $t \leq T$ *might not be needed to compute the process* MtM. *However, we are going to notice that the* MtM *process is involved in the equations for the others XVAs. In situations where we want to compute XVA processes without having computed first the MtM one, it will be useful to have the reduced equation for MtM at hand, so that everything can be computed under the same reduced stochastic basis* $(\mathbb{F}, \mathbb{P})$.

We now provide conditions under which the reduced equation (hence also the original one, by the previous theorem) admits a unique solution.

Theorem 3.5.4. *Assume that $\mathcal{Y}' \in \mathbb{S}_2'$, that the map $z \in \mathbb{R} \mapsto j_t(z - \mathcal{Y}_t')$ is almost surely Lipschitz in z, uniformly in t, and that $\mathbb{E}'\left[\int_0^T (j_t(-\mathcal{Y}_t'))^2\, dt\right] < +\infty$. Then the reduced value equation for $\mathcal{Y}$ and j admits a unique $\mathbb{S}_2'$ solution Z'. In particular, the shareholder value equation admits a unique $\mathbb{S}_2$ solution Z^{τ^-}.*

3.5.3 Economic Capital

As explained in the static case, capital requirements are focused on the solvency issue: Basel II Pilar II defines economic capital as the $\alpha = 99\%$ value at risk of depletion of core equity tier I capital over a year. The Fundamental review of the trading book required a shift from value at risk at level 99% to the expected shortfall at level 97.5%.

Here, CET1 depletion corresponds to the shareholder trading loss process $\mathcal{L}^{\tau^-}$, and these economic capital computations are made assuming that the bank does not default, hence under the probability $\mathbb{P}$:

$$
\begin{aligned}
\mathrm{EC}_t &= \mathbb{ES}'_{97.5\%,t}\left(\mathcal{L}'_{(t+1)\wedge T} - \mathcal{L}'_t\right) \\
&= \frac{1}{1 - 0.975} \int_{0.975}^{1} \mathrm{VaR}'_{\alpha,t}\left(\mathcal{L}'_{(t+1)\wedge T} - \mathcal{L}'_t\right) d\alpha,
\end{aligned}
$$

where $\mathrm{VaR}'_{\alpha,t} := \inf\{q \mid \mathbb{P}'_t[\mathcal{L} \geq q] \geq \alpha\}$.

Notice that since $\mathcal{L}'$ is a $(\mathbb{F}, \mathbb{P})$-local martingale, we are computing the expected shortfall of a centered variable, which is always a non-negative number.

Notice also that we introduced shareholder value and reduced value for $\mathcal{Y}$ and j, where j is a $\mathcal{P}(\mathbb{F}) \times \mathcal{B}(\mathbb{R})$-measurable. For the KVA process, we had

$$
\begin{aligned}
j : \Omega \times [0, T] \times \mathbb{R} &\to \mathbb{R}, \\
(\omega, t, z) &\mapsto h(\mathrm{EC}_t(\omega) - z)^+,
\end{aligned}
$$

which is indeed $\mathcal{P}(\mathbb{F}) \times \mathcal{B}(\mathbb{R})$-measurable as EC is $\mathbb{F}$-predictable.

3.6 XVAs for bilateral portfolios

We assume now that the bank is engaged in deals involving different clients (or at least *netting sets*, meaning that cash-flows are summed into one process), with only European derivatives. Let $\mathcal{C}$ be the finite set of clients. We define and study the different cash-flows involved in this situation, together with the associated MtM and XVA processes.

3.6.1 The promised cash-flows and the MtM process

We denote, for each client c in the finite set of clients $\mathscr{C}$, $\mathcal{P}^c$ the cumulative process of promised net cash-flows for the netting set c, i.e. from the client c to the bank, and τ_c

and R_c are the default time and the recovery rate of the client. We define $t_c := \tau^c \wedge \tau$, which is the default time corresponding to the netting set c.

We last define P^c as the value process for $\mathcal{P}^c$:

$$\begin{aligned} P^c_t &= 0, \quad t \geq T \\ P^c_t &= \mathbb{E}_t \left[\mathcal{P}^c_T - \mathcal{P}^c_t\right], \quad t \leq T, \end{aligned}$$

meaning that $\mathcal{P}^c + P^c$ is a $(\mathbb{G}, \mathbb{Q})$-martingale.

We assume that P^c is also the shareholder value for $\mathcal{P}^c$:

$$\begin{aligned} &P^c_t 1_{\{t \geq T\}} 1_{\{T < \tau\}} = 0, \\ &(P^c)^{\tau^-}_t = \mathbb{E}_t \left[(\mathcal{P}^c)^{\tau^-}_{\tau \wedge T} - (\mathcal{P}^c)^{\tau^-}_t + 1_{\{\tau \leq T\}} (P^c)^{\tau^-}_\tau \right), \quad t \leq \tau \wedge T. \end{aligned}$$

We assume that, in case of default, the liquidation is instantaneous. Moreover, we assume that no collateral is exchanged between the client and the bank.

In that situation, the promised cash-flows process is defined, on $[0, T]$, by

$$\mathcal{P} = \sum_{c \in \mathscr{C}} \left((\mathcal{P}^c)^{t_c} + 1_{[t_c, \infty)} P^c_{t_c}\right)$$

Indeed, for example in case of zero-recovery, we have the following decomposition,

$$\mathcal{P} = \sum_{c \in \mathscr{C}} \left((\mathcal{P}^c)^{t_c^-} + P^c_{(t_c)^-} 1_{[t_c, \infty)} + \left((\mathcal{P}^c)^{t_c} - (\mathcal{P}^c)^{t_c^-} + (P^c)^{t_c} - (P^c)^{t_c^-} \right) \right).$$

which can also be written dynamically:

$$d\mathcal{P}_t = \sum_{c \in \mathscr{C}} \left(1_{\{t < t_c\}} d\mathcal{P}^c_t + P^c_{t_c^-} \delta_{t_c}(dt) + \left(\mathcal{P}^c_{t_c} - \mathcal{P}^c_{t_c^-} + P^c_{t_c} - P^c_{t_c^-} \right) \delta_{t_c}(dt) \right).$$

Inside each netting set, the first term corresponds to the cash-flows exchanged between the client and the bank before the netting set default. The second term corresponds to the clean margin account becoming the property to the clean desk. The last term corresponds to the cash-flows exchanged between the CVA desk and the clean desk between the default and the liquidation, as the CVA desks compensates for the unpaid cash-flows and the mark-to-market fluctuations during the liquidation. The risk associated to the payments during the liquidation is called the gap risk.

Remark 3.6.1. *Notice that $\mathcal{P}$ is additive over individual trades, meaning that if $\mathcal{D}_c$ is the set of trades (deals) inside the netting set $c \in \mathcal{C}$ and for $d \in \mathcal{D}_c$, if $\mathcal{P}^d$ (resp P^d) is the cumulative process of cashflows (resp. the associated value process) for the trade d, we have*

$$\mathcal{P}^c = \sum_{d \in \mathcal{D}_c} \mathcal{P}^d$$

and

$$P_t^c = \mathbb{E}_t\left[\mathcal{P}_T^c - \mathcal{P}_t^c\right] = \sum_{d\in\mathcal{D}_c} \mathbb{E}_t\left[\mathcal{P}_T^d - \mathcal{P}_t^d\right] = \sum_{d\in\mathcal{D}_c} P_t^d,$$

thus, with $\mathcal{D} = \bigcup_{c\in\mathcal{C}} \mathcal{D}_c$ *the set of all deals and, for* $d \in \mathcal{D}_c$, $\tau_{c(d)} := \tau_c$ *and* $t_{c(d)} := t_c$,

$$\begin{aligned}\mathcal{P} &= \sum_{c\in\mathcal{C}} \sum_{d\in\mathcal{D}_c} \left(\left(\mathcal{P}^d\right)^{t_{c(d)}} + 1_{[t_{c(d)},\infty)} P^d_{t_{c(d)}}\right) \\ &= \sum_{d\in\mathcal{D}} \left(\left(\mathcal{P}^d\right)^{t_{c(d)}} + 1_{[t_{c(d)},\infty)} P^d_{t_{c(d)}}\right).\end{aligned}$$

We now compute the MtM process, which is the value process for $\mathcal{P}$.

Theorem 3.6.2. *We have, on* $[0, T]$,

$$\mathrm{MtM} = \sum_{c\in\mathscr{C}} P^c 1_{[0,t_c)}.$$

In addition, we have on $[0, T]$

$$\mathcal{P} + \mathrm{MtM} = \sum_{c\in\mathscr{C}} \left(\mathcal{P} + P\right)^{t_c},$$

which is a martingale, and MtM *is both the value process and the shareholder value process for* $\mathcal{P}$.

Proof. For $t \le T$, since MtM is the value process for $\mathcal{P}$, we have:

$$\begin{aligned}\mathrm{MtM}_t &= \mathbb{E}_t\left[\mathcal{P}_T - \mathcal{P}_t\right] \\ &= \sum_{c\in\mathscr{C}} \mathbb{E}_t\left[(\mathcal{P}^c)_T^{t_c} + 1_{[t_c,\infty)}(T) P^c_{t_c} - (\mathcal{P}^c)_t^{t_c} - 1_{[t_c,\infty)}(t) P^c_{t_c}\right] \\ &= \sum_{c\in\mathscr{C}} \mathbb{E}_t\left[\mathcal{P}^c_{T\wedge t_c} - \mathcal{P}^c_{t\wedge t_c} + 1_{\{t<t_c\le T\}} P^c_{t_c}\right] \\ &= \sum_{c\in\mathscr{C}} \mathbb{E}_t\left[\left(\mathcal{P}^c_{T\wedge t_c} - \mathcal{P}^c_{t\wedge t_c} + 1_{\{t<t_c\le T\}} P^c_{t_c}\right) 1_{\{t<t_c\}} + \left(\mathcal{P}^c_{T\wedge t_c} - \mathcal{P}^c_{t\wedge t_c} + 1_{\{t<t_c\le T\}} P^c_{t_c}\right) 1_{\{t_c\le t\}}\right].\end{aligned}$$

For fixed $c \in \mathscr{C}$, on $\{t_c \le t\}$, notice that $\mathcal{P}^c_{t\wedge t_c} = \mathcal{P}^c_t$, $\mathcal{P}^c_{T\wedge t_c} = \mathcal{P}^c_T$ as $t_c \le t \le T$, and $1_{\{t<t_c\le T\}} P^c_{t_c} = 0$, so the second term vanishes. On $\{t < t_c\} \in \mathcal{G}_t$, we have $\mathcal{P}^c_{t\wedge t_c} = \mathcal{P}^c_t$ and, since $P^c_T = 0$,

$$1_{\{t<t_c\le T\}} P^c_{t_c} = 1_{\{t_c\le T\}} P^c_{t_c} = 1_{\{t_c\le T\}} P^c_{t_c} + 1_{\{T<t_c\}} P^c_T = P^c_{T\wedge t_c}$$

thus we obtain

$$
\begin{aligned}
\mathrm{MtM}_t &= \sum_{c\in\mathscr{C}} \mathbb{E}_t\left[\left(\mathcal{P}^c_{T\wedge t_c} - \mathcal{P}^c_t + P^c_{T\wedge\tau^c}\right) 1_{\{t<t_c\}}\right] \\
&= \sum_{c\in\mathscr{C}} \left(\mathbb{E}_t\left[\mathcal{P}^c_T - \mathcal{P}^c_t\right] - \mathbb{E}_t\left[\mathcal{P}^c_T - \mathcal{P}^c_{T\wedge t_c}\right] + \mathbb{E}_t\left[P^c_{T\wedge t_c}\right]\right) 1_{\{t<t_c\}} \\
&= \sum_{c\in\mathscr{C}} \left(P^c_t - \mathbb{E}_t\left[\mathbb{E}_{T\wedge t_c}\left[\mathcal{P}^c_T - \mathcal{P}^c_{T\wedge t_c}\right]\right] + \mathbb{E}_t\left[P^c_{T\wedge t_c}\right]\right) 1_{\{t<t_c\}} \\
&= \sum_{c\in\mathscr{C}} \left(P^c_t - \mathbb{E}_t\left[P^c_{T\wedge t_c}\right] + \mathbb{E}_t\left[P^c_{T\wedge t_c}\right]\right) 1_{\{t<t_c\}} \\
&= \sum_{c\in\mathscr{C}} P^c_t 1_{\{t<t_c\}}.
\end{aligned}
$$

Now, we have

$$
\begin{aligned}
\mathcal{P} + \mathrm{MtM} &= \sum_{c\in\mathscr{C}} \left((\mathcal{P}^c)^{t_c} + 1_{[t_c,\infty)} P^c_{t_c} + 1_{[0,t_c)} P^c\right) \\
&= \sum_{c\in\mathscr{C}} \left((\mathcal{P}^c)^{t_c} + (P^c)^{t_c}\right) \\
&= \sum_{c\in\mathscr{C}} (\mathcal{P}^c + P^c)^{t_c}.
\end{aligned}
$$

Since P^c is also the shareholder value process for $\mathcal{P}^c$ for each netting set $c \in \mathscr{C}$, we have that $(\mathcal{P}^c + P^c)^{\tau^-}$ is a martingale on $[0, \tau \wedge T]$, so $\left((\mathcal{P}^c + P^c)^{t_c}\right)^{\tau^-} = \left((\mathcal{P}^c + P^c)^{\tau^-}\right)^{t_c}$ is also one, so their sum $(\mathcal{P} + \mathrm{MtM})^{\tau^-}$ is and MtM is the shareholder value process for $\mathcal{P}$. □

3.6.2 The counterparty cash-flows and the CVA process

Taking into account the defaults, the realized cash-flows from the clients to the bank are:

$$
\begin{aligned}
\mathcal{P} - \mathcal{C} = \sum_{c\in\mathcal{C}} \Bigg[(\mathcal{P}^c)^{t_c^-} + 1_{[t_c,\infty)} \Bigg(1_{\tau_c\le\tau} \left[R_c \left(P^c_{\tau_c} + \mathcal{P}^c_{\tau_c} - \mathcal{P}^c_{\tau_c^-}\right)^+ - \left(P^c_{\tau_c} + \mathcal{P}^c_{\tau_c} - \mathcal{P}^c_{\tau_c^-}\right)^- \right] \\
+ 1_{\tau<\tau_c} \left[(P^c_\tau + \mathcal{P}^c_\tau - \mathcal{P}^c_{\tau^-})^+ - R\,(P^c_\tau + \mathcal{P}^c_\tau - \mathcal{P}^c_{\tau^-})^- \right] \Bigg) \Bigg]
\end{aligned}
$$

One thus obtains, as $(\mathcal{P}^c)^{t_c} - (\mathcal{P}^c)^{t_c^-} = \left(\mathcal{P}^c_{t_c} - \mathcal{P}^c_{t_c^-}\right) 1_{[t_c,\infty)}$,

$$
\begin{aligned}
\mathcal{C} = \sum_{c\in\mathcal{C}} \Big[& (\mathcal{P}^c)^{t_c} - (\mathcal{P}^c)^{t_c^-} \\
& + 1_{[t_c,\infty)} \Big(1_{\{\tau_c \le \tau\}} \Big[P^c_{\tau_c} - R_c \left(P^c_{\tau_c} + \mathcal{P}^c_{\tau_c} - \mathcal{P}^c_{\tau_c^-} \right)^+ + \left(P^c_{\tau_c} + \mathcal{P}^c_{\tau_c} - \mathcal{P}^c_{\tau_c^-} \right)^- \Big] \\
& \qquad + 1_{\tau<\tau_c} \left[P^c_\tau - (P^c_\tau + \mathcal{P}^c_\tau - \mathcal{P}^c_{\tau-})^+ + R\,(P^c_\tau + \mathcal{P}^c_\tau - \mathcal{P}^c_{\tau-})^- \right] \Big) \Big] \\
= \sum_{c\in\mathcal{C}} \Big[& 1_{[t_c,\infty)} \Big(1_{\{\tau_c \le \tau\}} \Big[\mathcal{P}^c_{\tau_c} - \mathcal{P}^c_{\tau_c^-} + P^c_{\tau_c} - R_c \left(P^c_{\tau_c} + \mathcal{P}^c_{\tau_c} - \mathcal{P}^c_{\tau_c^-} \right)^+ + \left(P^c_{\tau_c} + \mathcal{P}^c_{\tau_c} - \mathcal{P}^c_{\tau_c^-} \right)^- \Big] \\
& \qquad + 1_{\tau<\tau_c} \left[\mathcal{P}^c_\tau - \mathcal{P}^c_{\tau-} + P^c_\tau - (P^c_\tau + \mathcal{P}^c_\tau - \mathcal{P}^c_{\tau-})^+ + R\,(P^c_\tau + \mathcal{P}^c_\tau - \mathcal{P}^c_{\tau-})^- \right] \Big) \Big].
\end{aligned}
$$

Using twice that $x = x^+ - x^-$ for all $x \in \mathbb{R}$, we obtain

$$
\mathcal{C} = \sum_{c\in\mathcal{C}} 1_{[t_c,\infty)} \left(1_{\{\tau_c \le \tau\}} (1-R_c) \left(P^c_{\tau_c} + \mathcal{P}^c_{\tau_c} - \mathcal{P}^c_{\tau_c^-} \right)^+ - 1_{\{\tau<\tau_c\}} (1-R) \left(P^c_\tau + \mathcal{P}^c_\tau - \mathcal{P}^c_{\tau-} \right)^- \right).
$$

While $\mathcal{P}$ is additive over individual trades, notice here that $\mathcal{C}$ is only additive over netting sets, since each summand is a nonlinear function of $\mathcal{P}^c$.

Notice that $\mathcal{C}^{\tau^-}$ is non-decreasing, according to our initial assumptions (as the bank has not defaulted yet, the bank only needs to compensate incoming flows from defaulted clients).

We now switch to the counterparty value adjustment process $\mathrm{CVA} = \mathrm{CVA}^{\tau^-} + {}^{\tau^-}\mathrm{CVA}$, which is defined by

$$
\begin{aligned}
\mathrm{CVA}^{\tau^-} &= \left(\mathrm{CVA}'\right)^{\tau^-}, \\
{}^{\tau^-}\mathrm{CVA} &= -\mathrm{CVA}_{\tau-} 1_{[\tau,+\infty)},
\end{aligned}
$$

where the second equation comes from our definition of shareholder value, which asks for $\mathrm{CVA}_t 1_{\{t\ge T\}} 1_{\{T<\tau\}} = 0$, and from our convention that $\mathrm{CVA}_t 1_{\tau\le t} 1_{\tau<T} = 0$.

Since $\tau' = +\infty$, the reduced process $\mathcal{C}'$ writes

$$
\mathcal{C}' = \sum_{c\in\mathcal{C}} 1_{[\tau_c',\infty)} (1-R_c) \left(P^c_{\tau_c'} + (\mathcal{P}^c)'_{\tau_c'} - (\mathcal{P}^c)'_{(\tau_c')-} \right)^+ .
$$

This is a non-decreasing process on $[0,T]$, so it is in $\mathbb{S}_2'$ if and only if $\mathcal{C}_T'$ is square integrable. We then obtain the following theorem, in application to Theorem 3.5.4.

Proposition 3.6.3. *Assume that* $\mathbb{E}'\left[(\mathcal{C}'_T)^2\right] < +\infty$.
Then $\mathcal{C}' \in \mathbb{S}'_2$ *and a reduced value for* $\mathcal{C}$ *and* $j = 0$ *exists uniquely, meaning that the following system admits a unique solution:*

$$\mathrm{CVA}'_T = 0,$$
$$\mathrm{CVA}'_t = \mathbb{E}'_t\left[\mathcal{C}'_T - \mathcal{C}'_t\right] = \mathbb{E}'_t\left[\sum_{c\in\mathcal{C}} 1_{t<\tau'_c\le T}(1-R_c)\left(P^c_{\tau'_c} + (\mathcal{P}^c)'_{\tau'_c} - (\mathcal{P}^c)'_{(\tau'_c)-}\right)^+\right], \quad t \le T.$$

Of course, the resolution to this equation is much more involved than the previous one. As mentionned above, by linearity, one can consider one equation for each netting set $c \in \mathcal{C}$, but one has then to numerically compute the conditional expectations.

3.6.3 The funding cash-flows and the FVA process

We now tackle the funding problem and its associated cost. In contrast with the previous definitions and computations, which were more or less canonical, the situation here depends on the funding strategy of the bank, and of what accounts can be used for funding purposes.

It depends also on the risky funding rate at which the bank can borrow money from external funders. We define the risky funding asset price as the solution to the following equation:

$$U_0 = 1,$$
$$dU_t = \lambda_t U_t dt + (1-R)U_{t-}dJ_t, \quad t \le \tau \wedge T,$$

where we recall that J is the survival process of the bank, defined as $J = 1_{[0,\tau)}$, where R is the (constant) recovery rate of the bank, and where λ is a non-negative predictable process representing the unsecured borrowing rate. If the bank borrows 1 at time $t = 0$, then U_t is the amount it has to give back to external funders at time t. Since $dJ_t = -\delta_\tau(dt)$ where δ denotes the Dirac mass, we have

$$\begin{aligned} U_t 1_{\{t<\tau\}}1_{\{t\le T\}} &= \left(1 + \int_0^t \lambda_s U_s ds\right)1_{\{t<\tau\}}1_{\{t\le T\}} - (1-R)\int_0^t U_{s-}\delta_\tau(ds)1_{\{t<\tau\}}1_{\{t\le T\}} \\ &= \left(1 + \int_0^t \lambda_s U_s ds\right)1_{\{t<\tau\}}1_{\{t\le T\}}. \end{aligned}$$

Thus U solves $dU_t = \lambda_t U_t dt$ on $[0,\tau) \cap [0,T]$, and we obtain

$$U_t 1_{\{t<\tau\}}1_{\{t\le T\}} = e^{\int_0^t \lambda_s ds}1_{\{t<\tau\}}1_{\{t\le T\}},$$

thus $U_{\tau^-}1_{\{\tau\leq T\}} = ue^{\int_0^\tau \lambda_s}$ and

$$\begin{aligned}
U_\tau 1_{\{\tau\leq T\}} &= \left(U_{\tau^-} + \int_{\tau^-}^{\tau} dU_t\right) 1_{\{\tau\leq T\}} \\
&= \left(U_{\tau^-} + \int_{\tau^-}^{\tau} \lambda_t U_t dt - (1-R)\int_{\tau^-}^{\tau} U_{t^-}\delta_\tau(dt)\right) 1_{\{\tau\leq T\}} \\
&= \left(U_{\tau^-} - (1-R)U_{\tau^-}\right) 1_{\{\tau\leq T\}} \\
&= RU_{\tau^-}1_{\{\tau\leq T\}}.
\end{aligned}$$

consistently with the recovery of the bank towards external funders.

Note that the SDE for U is equivalent to

$$\begin{aligned}
&U_0^{\tau^-} = 1, \\
&dU_t^{\tau^-} = J_t\lambda_t U_t dt, \quad t\geq 0, \\
&U_\tau 1_{\{\tau\leq T\}} = RU_{\tau^-}1_{\{\tau\leq T\}}
\end{aligned}$$

We assume that the capital at risk is not used by the bank for its funding purposes. In particular, the only funding source is the reserve capital account. This account is marked-to-model and thus satisfied $\mathrm{RC}_t = \mathrm{CA}_t$ for all $t \geq 0$. Since the CA desks has to post collateral in the clean margin account, which is also market-to-model and satisfies $\mathrm{CM}_t = \mathrm{MtM}_t$ for all $t \geq 0$. Thus the amount needed to be borrowed from external funders at each time is $(\mathrm{MtM}_t - \mathrm{CA}_t)^+ = \alpha_t U_t$ for α_t the number of risky funding asset invested in at time t.

Since the strategy is self-financing, we find that the cumulated risky funding cash flows $\mathcal{F}$ has dynamics

$$\begin{aligned}
\mathcal{F}_0 &= 0, \\
d\mathcal{F}_t &= \alpha_{t^-}dU_t = \alpha_t\lambda_t U_t dt + (1-R)\alpha_{t^-}U_{t^-}dJ_t \\
&= \lambda_t(\mathrm{MtM}_t - \mathrm{CA}_t)^+ dt + (1-R)(\mathrm{MtM}_{t^-} - \mathrm{CA}_{t^-})^+ dJ_t.
\end{aligned}$$

This writes, as $\mathrm{CA} = \mathrm{CVA} + \mathrm{FVA}$,

$$\begin{aligned}
&\mathcal{F}^{\tau^-} = 0, \\
&d\mathcal{F}_t^{\tau^-} = J_t\lambda_t(\mathrm{MtM}_t - \mathrm{CVA}_t - \mathrm{FVA}_t^{\tau^-})^+ dt, \\
&\mathcal{F}_\tau 1_{\{\tau\leq T\}} = \left(\mathcal{F}_{\tau^-} - (1-R)(\mathrm{MtM}_{\tau^-} - \mathrm{CVA}_{\tau^-} - \mathrm{FVA}_{\tau^-})^+\right) 1_{\{\tau\leq T\}}
\end{aligned}$$

We now switch to the funding value adjustment process $\mathrm{FVA} = \mathrm{FVA}^{\tau^-} + {}^{\tau^-}\mathrm{FVA}$, which is defined by

$$\begin{aligned}
\mathrm{FVA}^{\tau^-} &= \left(\mathrm{FVA}'\right)^{\tau^-}, \\
{}^{\tau^-}\mathrm{FVA} &= -\mathrm{FVA}_{\tau^-}1_{[\tau,+\infty)},
\end{aligned}$$

where the second equation is obtained in a similar way as for the CVA, see above.

The shareholder value equation for $\mathcal{F}$ on $(\mathbb{G}, \mathbb{Q})$ thus writes:

$$
\begin{aligned}
\mathrm{FVA}_t^{\tau^-} &= \mathbb{E}_t\left[\mathcal{F}_{\tau\wedge T}^{\tau^-} - \mathcal{F}_t^{\tau^-} + 1_{\{\tau\le T\}}\mathrm{FVA}_{\tau^-}\right] \\
&= \mathbb{E}_t\left[\int_t^{\tau\wedge T} \lambda_s \left(\mathrm{MtM}_s - \mathrm{CVA}_s - \mathrm{FVA}_s^{\tau^-}\right)^+ ds + 1_{\{\tau\le T\}}\mathrm{FVA}_{\tau^-}\right] \\
&= \mathbb{E}_t\left[\int_t^{\tau\wedge T} \lambda'_s \left(\mathrm{MtM}'_s - \mathrm{CVA}'_s - \mathrm{FVA}_s^{\tau^-}\right)^+ ds + 1_{\{\tau\le T\}}\mathrm{FVA}_{\tau^-}\right],
\end{aligned}
$$

as, for $t \le s < \tau \wedge T$, $\lambda_s = \lambda_s^{\tau^-} = (\lambda')_s^{\tau^-} = \lambda'_s$, and the same is true for MtM and CVA.

We recognize a shareholder value equation for $\mathcal{Y} = 0$ and j defined by

$$
\begin{aligned}
j : \Omega \times [0,T] \times \mathbb{R} &\to \mathbb{R} \\
(\omega, t, z) &\mapsto \lambda'_s(\omega)\left(\mathrm{MtM}'_s(\omega) - \mathrm{CVA}'_s(\omega) - z\right)^+.
\end{aligned}
$$

The associated reduced value equation is:

$$
\begin{aligned}
&\mathrm{FVA}'_T = 0, \\
&\mathrm{FVA}'_t = \mathbb{E}'_t\left[\int_t^T \lambda'_s \left(\mathrm{MtM}'_s - \mathrm{CVA}'_s - \mathrm{FVA}'_s\right)^+ ds\right], \quad t \le T.
\end{aligned}
$$

To show that FVA^{τ^-} is well-defined, it is now enough to show that the previous equation has a unique solution. From Theorem 3.5.4, since $z \mapsto j_t(z)$ is almost surely Lipschitz-continuous, it is enough to have that $z \mapsto j_t(z)$ is almost surely Lipschitz-continuous, uniformly in t, and that $\mathbb{E}'\left[\int_0^T (j_t(0))^2\, dt\right] < +\infty$.

3.6.4 The KVA process

Remember that the capital value adjustment process $\mathrm{KVA} = \mathrm{KVA}^{\tau^-} + {}^{\tau^-}\mathrm{KVA}$ is defined by

$$
\begin{aligned}
\mathrm{KVA}^{\tau^-} &= \left(\mathrm{KVA}'\right)^{\tau^-}, \\
{}^{\tau^-}\mathrm{KVA} &= -\mathrm{KVA}_{\tau^-}1_{[\tau,\infty)},
\end{aligned}
$$

where KVA' is the $\mathbb{F}$-reduction of KVA, solution to

$$
\begin{aligned}
&\mathrm{KVA}'_T = 0, \\
&\mathrm{KVA}'_t = \mathbb{E}'_t\left[\int_t^T h\left(\mathrm{EC}_s - \mathrm{KVA}'_s\right)^+ ds\right], \quad t \le t,
\end{aligned}
$$

and where EC is defined by

$$
\mathrm{EC}_t = \mathbb{ES}'_{97.5\%,t}\left(\mathcal{L}'_{(t+1)\wedge T} - \mathcal{L}'_t\right), \quad t \le T.
$$

Proposition 3.6.4. *Assume that* $\mathbb{E}'\left[\int_0^T (\mathcal{L}'_t)^2\,dt\right] < +\infty$.

Then $\mathbb{E}'\left[\int_0^T (\mathrm{EC}_t)^2\,dt\right] < +\infty$ *and* KVA' *is well defined in* $\mathbb{S}'_2$. *Moreover, we have, for all* $0 \le t \le T$,

$$\begin{aligned}\mathrm{KVA}'_t &= h\mathbb{E}'_t\left[\int_t^T e^{-h(s-t)}\max(\mathrm{EC}_s, \mathrm{KVA}'_s)ds\right]\\ &= h\mathbb{E}'_t\left[\int_t^T e^{-h(s-t)}\mathrm{CR}'_s ds\right].\end{aligned}$$

Proof. First, it is well-known that $\mathbb{ES}'_{\alpha,t}$ is $\frac{1}{1-\alpha}$-Lipschitz continuous. Thus we obtain

$$\begin{aligned}|\mathrm{EC}_t| &= \left|\mathbb{ES}'_{\alpha,t}\left(\mathcal{L}'_{(t+1)\wedge T} - \mathcal{L}'_t\right) - \mathbb{ES}'_{\alpha,t}(0)\right|\\ &\le \frac{1}{1-\alpha}\left|\mathcal{L}'_{(t+1)\wedge T} - \mathcal{L}'_t\right|\\ &\le \frac{1}{1-\alpha}\left|\mathcal{L}'_{(t+1)\wedge T}\right| + \frac{1}{1-\alpha}\left|\mathcal{L}'_t\right|.\end{aligned}$$

Then, using $(a+b)^2 \le 2a^2 + 2b^2$,

$$\begin{aligned}\mathbb{E}'\left[\int_0^T (\mathrm{EC}_t)^2 dt\right] &\le \frac{2}{1-\alpha}\left(\mathbb{E}'\left[\int_0^T \left|\mathcal{L}'_{(t+1)\wedge T}\right|^2 dt\right] + \mathbb{E}'\left[\int_0^T \left|\mathcal{L}'_t\right|^2 dt\right]\right)\\ &\le \frac{4}{1-\alpha}\mathbb{E}'\left[\int_0^T \left|\mathcal{L}'_t\right|^2 dt\right] < +\infty.\end{aligned}$$

Now, we recognize that KVA' is the reduced value associated to $\mathcal{Y} = 0$ and j defined by

$$\begin{aligned}j : \Omega \times [0,T] \times \mathbb{R} &\to \mathbb{R},\\ (\omega, t, z) &\mapsto h\left(\mathrm{EC}_t(\omega) - z\right)^+.\end{aligned}$$

By Theorem 3.5.4, since $z \mapsto j_t(z)$ is clearly Lipschitz continuous and

$$\begin{aligned}\mathbb{E}'\left[\int_0^T (j_t(0))^2\,dt\right] &= \mathbb{E}'\left[\int_0^T h^2\,(\mathrm{EC}_t)^2\,dt\right]\\ &= h^2\mathbb{E}'\left[\int_0^T (\mathrm{EC}_t)^2\,dt\right] < +\infty,\end{aligned}$$

we obtain that KVA' is uniquely defined in $\mathbb{S}'_2$.

The equation for KVA' writes, as $(x-y)^+ = \max(x,y) - y$,

$$\mathrm{KVA}'_t = \mathbb{E}'_t\left[\int_t^T h(\mathcal{Y}'_s - \mathrm{KVA}'_s)ds\right], \quad t \le T,$$

with $\mathcal{Y}'_t = \max(\mathrm{EC}_t, \mathrm{KVA}'_t)$, $t \le T$. We have

$$\mathrm{KVA}'_t + \int_0^t h\left(\mathcal{Y}'_s - \mathrm{KVA}'_s\right) ds = \mathbb{E}'_t\left[\int_0^T \left(\mathcal{Y}'_s - \mathrm{KVA}'_s\right) ds\right],$$

hence KVA' solves, for a $(\mathbb{F},\mathbb{P})$-martingale μ,

$$\begin{aligned}&\mathrm{KVA}'_T = 0,\\ &d\mathrm{KVA}'_t - h\mathrm{KVA}'_t dt + h\mathcal{Y}'_t dt = d\mu_t.\end{aligned}$$

We set $\widetilde{\mathrm{KVA}}'_t = e^{-ht}\mathrm{KVA}'_t$ and $\widetilde{\mathcal{Y}}'_t = e^{-ht}\mathcal{Y}'_t$, and we have $d\widetilde{\mathrm{KVA}}'_t = e^{-ht}d\mathrm{KVA}'_t - he^{-ht}\mathrm{KVA}'_t dt$, so we obtain $\mathrm{K\tilde{V}A}'_T = 0$ and

$$\begin{aligned}e^{-ht}d\mu_t &= e^{-ht}d\mathrm{KVA}'_t - he^{-ht}\mathrm{KVA}'_t dt + he^{-ht}\mathcal{Y}'_t dt\\ &= d\widetilde{\mathrm{KVA}}'_t + h\widetilde{\mathcal{Y}}'_t dt.\end{aligned}$$

Thus $\widetilde{\mathrm{KVA}}' + h\int_0^{\cdot}\widetilde{\mathcal{Y}}'_s ds$ is a $(\mathbb{F},\mathbb{P})$-martingale, and we obtain

$$\widetilde{\mathrm{KVA}}'_t + h\int_0^t \widetilde{\mathcal{Y}}'_s ds = \mathbb{E}'_t\left[\widetilde{\mathrm{KVA}}'_T + \int_0^T \widetilde{\mathcal{Y}}'_s ds\right],$$

which gives

$$\widetilde{\mathrm{KVA}}'_t = \mathbb{E}'_t\left[\int_t^T \widetilde{\mathcal{Y}}'_s ds\right],$$

which is the announced equality, multiplying by e^{ht} on both sides. □

Chapter 4

Extensions and Numerics

In this section, we provide the extensions of our previous results with

- Collateral exchange between the bank and its clients,
- Capital at risk as a funding source.

We stay in the static setting for simplicity, all our results can (and should) be adapted to the continuous setting.

4.1 Collateral

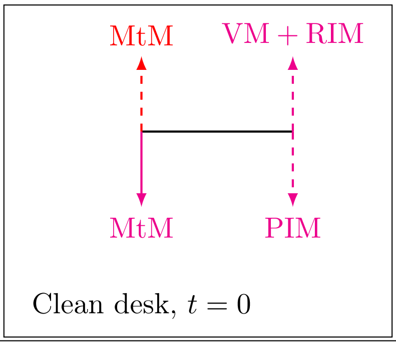
As explained in the Introduction, collateral is exchanged between between the bank and its clients in order to reduce one's exposure in case of default. Two types of collateral is exchanged:

- Variation Margin, which goal is to track the MtM value of the portfolio. It is typically re-hypothecable.
- Initial Margin, which goal is to reduce the gap risk and which is not fungible across deals.

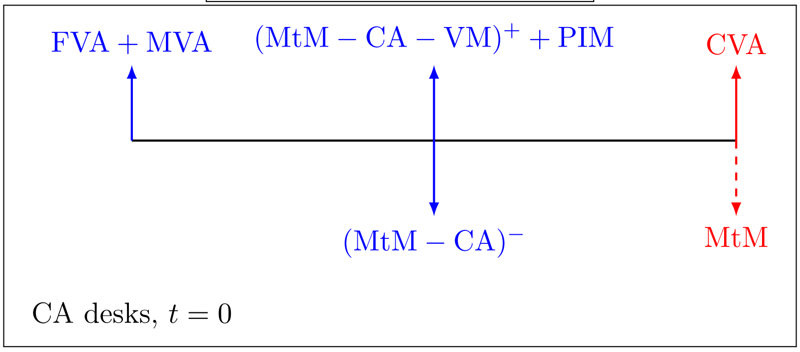
The collateral is here to guarantee some value in case of default of one party: in principle, it will induce a smaller CVA.

Let us denote VM the variation margin posted by the client (if positive, a negative VM means that it is posted by the bank) at time $t = 0$, and RIM (resp. PIM) the recieved Initial Margin, i.e. posted by the client (resp. the posted Initial Margin, posted by the bank). Since the Initial Margin is not fungible, it has to be borrowed separatly from the rest of the collateral, and this additional borrowing, gives rise to an additional expected cost (a contra-asset) which is incorporated with an additional value adjustment (MVA, Margin Value Adjustment). We now have $\mathrm{CA} = \mathrm{CVA} + \mathrm{FVA} + \mathrm{MVA}$.

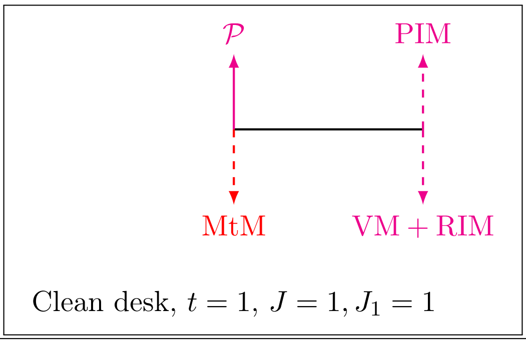
Under these hypothesis, we have the following cash flows:



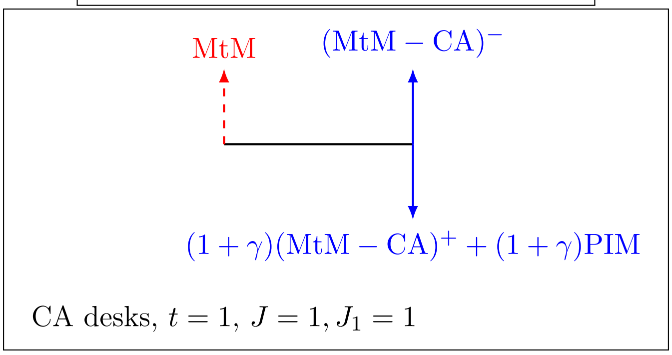
Clean desk, $t = 0$



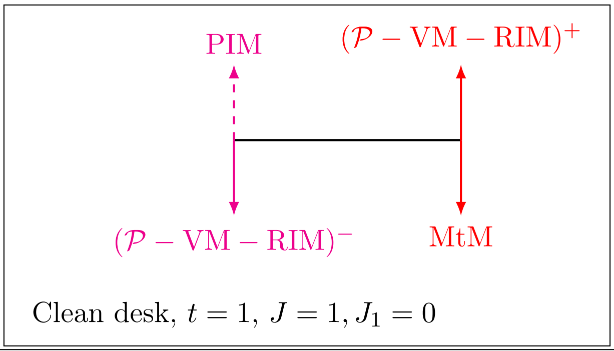
CA desks, $t = 0$



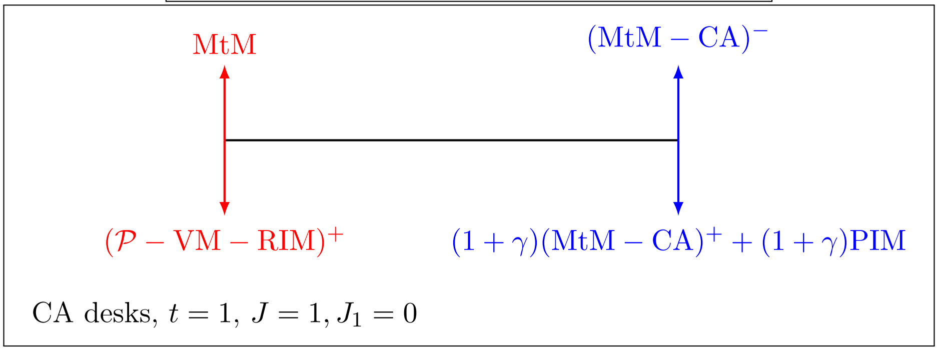
Clean desk, $t = 1,\ J = 1, J_1 = 1$



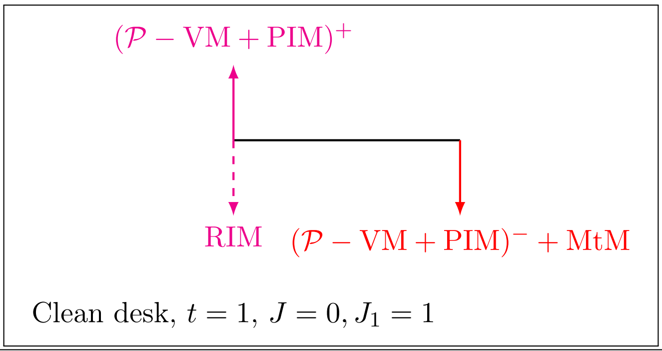
CA desks, $t = 1,\ J = 1, J_1 = 1$



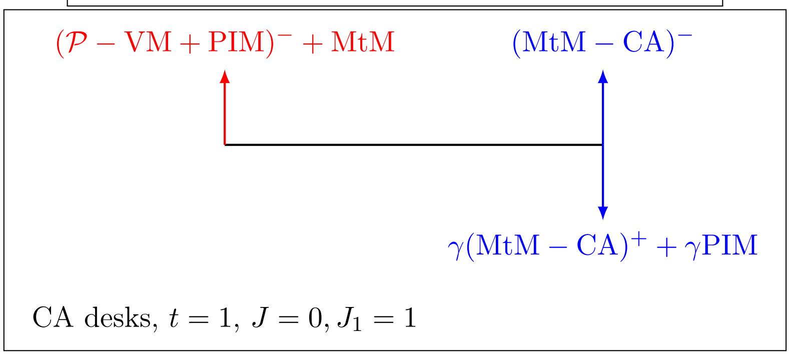
Clean desk, $t = 1,\ J = 1, J_1 = 0$



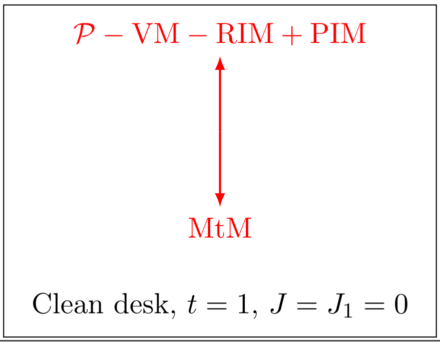
CA desks, $t = 1,\ J = 1, J_1 = 0$

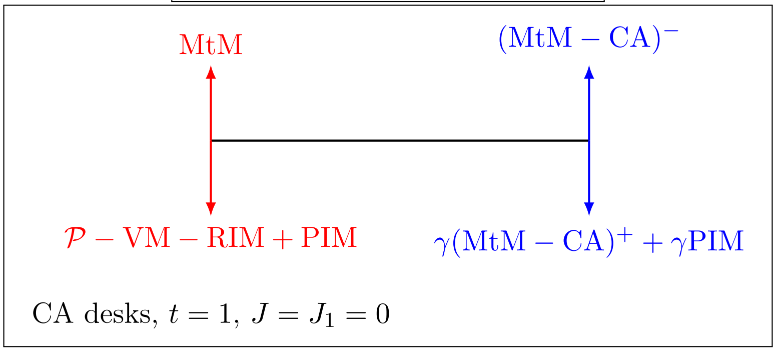


Clean desk, $t = 1,\ J = 0, J_1 = 1$



CA desks, $t = 1,\ J = 0, J_1 = 1$





With these pictures in mind, doing the same computations as in the second chapter, it is not difficult to observe that:

$$\begin{aligned}
\mathrm{CVA} &= \mathbb{E}^\star\left[J(1-J_1)(\mathcal{P}-\mathrm{VM}-\mathrm{RIM})^+ + (1-J)\mathrm{CVA}\right] = \mathbb{E}\left[(1-J_1)(\mathcal{P}-\mathrm{VM}-\mathrm{RIM})^+\right],\\
\mathrm{FVA} &= \mathbb{E}^\star\left[J\gamma(\mathrm{MtM}-\mathrm{VM}-\mathrm{CA})^+ + (1-J)\mathrm{FVA}\right]\\
&= \mathbb{E}^\star\left[J\gamma(\mathrm{MtM}-\mathrm{VM}-\mathrm{CVA}-\mathrm{FVA}-\mathrm{MVA})^+ + (1-J)\mathrm{FVA}\right]\\
&= \mathbb{E}\left[\gamma(\mathrm{MtM}-\mathrm{VM}-\mathrm{CVA}-\mathrm{FVA}-\mathrm{MVA})^+\right]\\
&= \frac{\gamma}{1+\gamma}(\mathrm{MtM}-\mathrm{VM}-\mathrm{CVA}-\mathrm{MVA})^+,\\
\mathrm{MVA} &= \mathbb{E}^\star\left[J\gamma\mathrm{PIM} + (1-J)\mathrm{MVA}\right] = \gamma\mathrm{PIM}.
\end{aligned}$$

The PIM and RIM are computed as $\mathbb{Q}$-value at risks of $\pm(\mathcal{P}-\mathrm{VM})$.

4.2 Capital at risk as a funding source

We now want to take into account the possibility to use the capital at risk as a funding source. It is interesting as the bank needs to borrow less money from the external funders, hence reducing the FVA.

When entering the deal, the bank recieves CVA, FVA and KVA from the client, and since the capital at risk is now a funding source, it only needs to borrow $(\mathrm{MtM}-\mathrm{VM}-\mathrm{CA}-\max(\mathrm{EC},\mathrm{KVA}))^+$, instead of $(\mathrm{MtM}-\mathrm{VM}-\mathrm{CA})^+$ as we had before.

The CVA and the MVA are unchanged, but we now obtain a *system* for the random variable L° and the FVA and KVA:

$$L^\circ = \mathcal{C}^\circ + \mathcal{F}^\circ_{VM} + \mathcal{F}^\circ_{IM} - J\mathrm{CA}$$
$$J(1-J_1)(\mathcal{P} - \mathrm{VM} - \mathrm{RIM})^+ + J\gamma(\mathrm{MtM} - \mathrm{VM} - \mathrm{CA} - \max(\mathrm{EC}, \mathrm{KVA}))^+ + J\gamma\mathrm{PIM} - J\mathrm{CA},$$

and

$$\begin{aligned}
\mathrm{FVA} &= \frac{\gamma}{1+\gamma}(\mathrm{MtM} - \mathrm{VM} - \mathrm{CVA} - \mathrm{MVA} - \max(\mathrm{EC}, \mathrm{KVA}))^+, \\
\mathrm{KVA} &= \frac{h}{1+h}\mathbb{ES}(L^\circ).
\end{aligned}$$

While the CVA and MVA are computed as before, the previous system can only be solved numerically, by Picard iteration, starting for example with $L^{(0)} = \mathrm{KVA}^{(0)} = 0$ and $\mathrm{FVA}^{(0)} = \frac{\gamma}{1+\gamma}(\mathrm{MtM} - \mathrm{VM} - \mathrm{CVA} - \mathrm{MVA})^+$, and then iterating the previous system upon convergence.

4.3 Continuous time

In continuous time, we directly work in the reduced probability space $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$, and we forgt the prime notation for simplicity. Assuming collateral exchanged between each client and the bank and assuming that the capital at risk is a funding source, we have $\mathrm{CA} = \mathrm{CVA} + \mathrm{FVA} + \mathrm{MVA}$ we obtain the following equations.

$$\begin{aligned}
\mathrm{CVA}_t &= \sum_{t<\tau_c} \mathbb{E}_t\left[(1-R_c)\left(P^c_{\tau_c} + \mathcal{P}^c_{\tau_c} - \mathcal{P}^c_{\tau_c^-} - \mathrm{VM}^c_{\tau_c^-} - \mathrm{RIM}^c_{\tau_c^-}\right)^+\right], \\
\mathrm{FVA}_t &= \mathbb{E}_t\left[\int_t^T \lambda_s \left(\sum_c J^c(P^c - \mathrm{VM}^c) - \mathrm{CA} - \max(\mathrm{EC}, \mathrm{KVA})\right)^+_s ds\right], \\
\mathrm{MVA}_t &= \mathbb{E}_t\left[\int_t^T \lambda_s \sum_c J^c_s \mathrm{PIM}^c_s ds\right], \\
\mathcal{L}_t &= \sum_{c:t\geq\tau_c} (1-R_c)\left(P^c_{\tau_c} + \mathcal{P}^c_{\tau_c} - \mathcal{P}^c_{\tau_c^-} - \mathrm{VM}^c_{\tau_c^-} - \mathrm{RIM}^c_{\tau_c^-}\right)^+ \\
&\quad + \int_0^t \lambda_s \left(\sum_c J^c(P^c - \mathrm{VM}^c) - \mathrm{CA} - \max(\mathrm{EC}, \mathrm{KVA})\right)^+_s ds \\
&\quad + \int_0^t \lambda_s \sum_c J^c_s \mathrm{PIM}^c_s ds + \mathrm{CA}_t - \mathrm{CA}_0, \\
\mathrm{EC}_t &= \mathbb{ES}_{t,0.975}\left(\mathcal{L}_{t+1\wedge T} - \mathcal{L}_t\right), \\
\mathrm{KVA}_t &= \mathbb{E}_t\left[\int_t^T h(\max(\mathrm{EC}_s - \mathrm{KVA}_s)^+ ds\right] = \mathbb{E}_t\left[\int_t^T he^{-h(s-t)}\max(\mathrm{EC}_s, \mathrm{KVA}_s) ds\right].
\end{aligned}$$

As in the static case, there is a forward-backward coupling between the forward loss process $\mathcal{L}$ and the FVA and KVA backward processes, through the economic capital EC. However, one can prove that this set of equations is well-posed, and a Picard algorithm converges to the solution of the above system.

4.4 A collaterallization scheme

4.4.1 A credit model

We define a credit model for the default of the counterparties and of the bank. We introduce a common shock model, where defauts can happen simulataneously with positive probability.

Remark 4.4.1. *Although the bank default time is dealt with the reduction of filtration framework that we studied in the third chapter, the spread of the bank still appears in the equations for the* FVA.

Let $n \geq 1$ be the number of clients and let each $1 \leq i \leq n$ represent one client, and let 0 represent the bank itself. Let $\mathcal{E} \subset \mathcal{P}(\{0,\ldots,n\})$ be a family of subsets of $\{0,\ldots,n\}$. For each subset $E \in \mathcal{E}$, let τ_E model the time when all the (non-defaulted names among) members $i \in E$ default. Each τ_E is modeled as an independent time-inhomogeneous exponential random variable with intensity function γ_E. For each $0 \leq i \leq n$, we then set $\tau_i := \min_{E\in\mathcal{E}:i\in E} \tau_E$. Consequently, the default intensity of τ_i is $\gamma^i := \sum_{E\in\mathcal{E}:i\in E} \gamma_E$.

Example 4.4.2. *If* $\mathcal{E} = \{\{0\},\{1\},\{2\},\{3\},\{4\},\{5\},\{1,3\},\{2,3\},\{0,1,2,4,5\}\}$*, one can for example observe the following sequence of defaults:*

- *At time* $t = 0.9$*, the event* $\{3\}$ *realize. Then* $\tau_3 = 0.9$.
- *At time* $t = 1.4$*, the event* $\{5\}$ *realize. Then* $\tau_5 = 1.4$.
- *At time* $t = 2.6$*, the event* $\{1,3\}$ *realize. Then, since* 3 *already defaulted, only* 1 *defaults at that time and* $\tau_1 = 2.6$.
- *At time* $t = 5.5$*, the event* $\{0,1,2,4,5\}$ *realize. As* 1 *and* 5 *have already defaulted, we have* $\tau_0 = \tau_2 = \tau_4 = 5.5$.

4.4.2 Collateral

We assume here that the previous common shock default model is in force. We assume that the portfolio of deals involving each clients is "fully collateralized", in the sense that $\mathrm{VM}^c_t = P^c_t$ for each $t \leq \tau_c$. We moreover assume that the initial margin are computed as Value at Risk:

$$\mathrm{RIM}^c_t = \mathrm{VaR}_{t,a_{rim}}\left(P^c_t + \mathcal{P}^c_t - P^c_{t^-} - \mathcal{P}^c_{t^-}\right), \mathrm{PIM}^c_t = \mathrm{VaR}_{t,a_{pim}}\left(-P^c_t - \mathcal{P}^c_t + P^c_{t^-} + \mathcal{P}^c_{t^-}\right).$$

Then, one obtains

$$\mathrm{CVA}_t = \sum_c 1_{t<\tau_c}(1-R_c)(1-a_{rim})\mathbb{E}_t\left[\int_t^T (\mathbb{ES}_s - \mathrm{VaR}_{s,a_{rim}})(P_s^c + \mathcal{P}_s^c - P_{s-}^c - \mathcal{P}_{s-}^c)\gamma_s^c e^{-\int_t^s \gamma_u^c du} ds\right].$$

Indeed, this comes from properties of the common shock model, plus the fact that

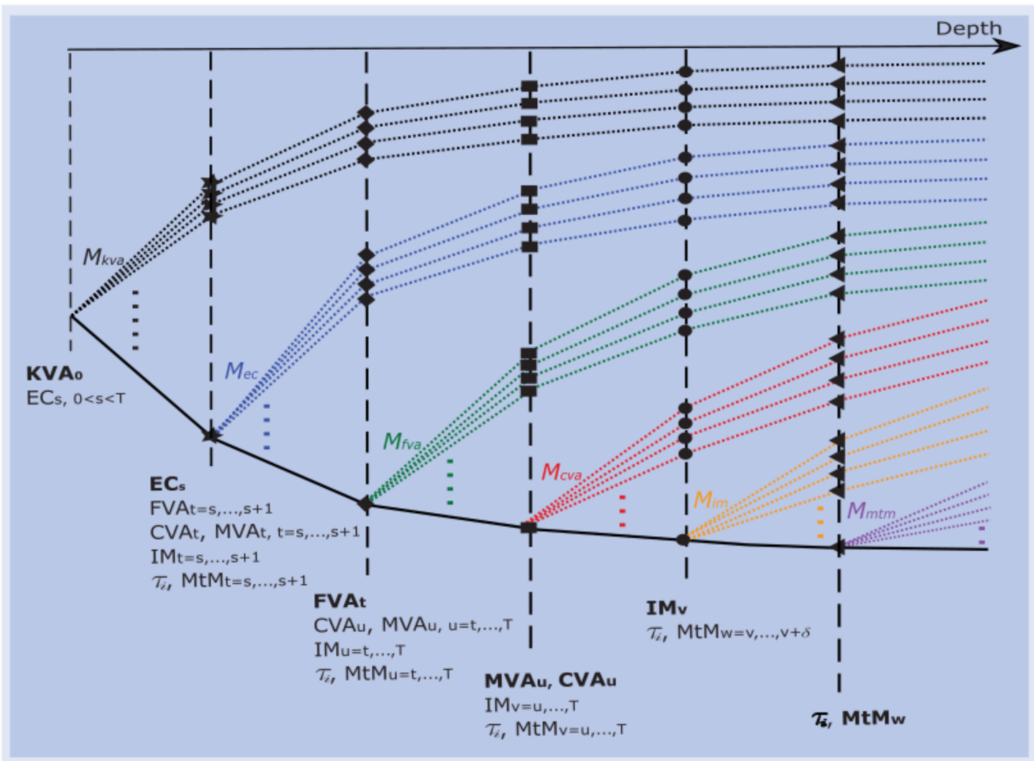
$$(P_t^c + \mathcal{P}_t^c - \mathcal{P}_{t-}^c - \mathrm{VM}_{t-}^c - \mathrm{RIM}_{t-}^c)^+ = (P_t^c - P_{t-}^c + \mathcal{P}_t^c - \mathcal{P}_{t-}^c - \mathrm{RIM}_{t-}^c)^+$$

which is of the form $(X - \mathrm{VaR}_\alpha(X))^+$. We have, assuming that X has a density for example,

$$\begin{aligned}\mathbb{E}\left[(X - \mathrm{VaR}_\alpha(X))^+\right] &= \mathbb{E}\left[(X - \mathrm{VaR}_\alpha(X))1_{X \geq \mathrm{VaR}_\alpha(X)}\right] \\ &= \mathbb{E}\left[X1_{X \geq \mathrm{VaR}_\alpha(X)}\right] - \mathrm{VaR}_\alpha(X)\mathbb{P}\left[X \geq \mathrm{VaR}_\alpha(X)\right] \\ &= (1-\alpha)\left(\mathbb{ES}_\alpha(X) - \mathrm{VaR}_\alpha(X)\right).\end{aligned}$$

4.5 Numerics

The dependence tree among the XVAs is represented graphically in the picture.



Note that, in this picture, we consider that the capital at risk is not a funding source, as some dependency of the KVA on the FVA would appear in that case. In fact, using

Picard iterations as explained above, one can disantangle this dependency and get a dependence tree as above.

We give two strategies for the numerical approximation of the XVAs:

- Nested simulations
- Deep learning

4.5.1 Nested Simulations

The algorithm

Thanks to the dependence tree, we observe that the XVA computations is decomposed into layers. In a *nested simulations* framework, the left layers are launched first and trigger on-the-fly nested ones, whenever needed.

For simplicity, let us write the dependences in functional form, for example, we write CVA(MtM) and FVA(CVA(MtM)) to emphasize on the fact that the CVA is a functional of the MtM process. We have, taking only one client and no collateral for simplicity,

$$\mathrm{CVA}_0 = \mathbb{E}\left[(1-R)(\mathrm{MtM}_{\tau_c} + \mathcal{P}_{\tau_c} - \mathcal{P}_{\tau_c^-})^+\right]$$

This is computed using a Monte-Carlo approximation:

$$\mathrm{CVA}_0^N = \frac{1-R}{N_{\mathrm{CVA}}} \sum_{i=1}^{N_{\mathrm{CVA}}} \left(\mathrm{MtM}^i_{\tau_c^i} + \Delta\mathcal{P}^i_{\tau_c^i}\right)^+,$$

where $(\mathcal{P}^i, \tau_c^i)$ are i.i.d. realizations of τ_c. Notice that here, we still need to compute each $\mathrm{MtM}_{\tau_c^i}$, which is defined as the conditional expectation, conditionnally to the ith risk-factors underlying the simulation simulated up to τ_c^i

$$\mathrm{MtM}^i_{\tau_c^i} = \mathbb{E}_i\left[\mathcal{P}_T - \mathcal{P}_{\tau_c^i}\right].$$

Then, for each $1 \leq i \leq N_{\mathrm{CVA}}$, we need to compute the following "nested" Monte-Carlo Approximation:

$$\mathrm{MtM}^i_{\tau_c^i} = \frac{1}{N_{\mathrm{MtM}}} \sum_{j=1}^{N_{\mathrm{MtM}}} (\mathcal{P}^{i,j}_T - \mathcal{P}^{i,j}_{\tau_c^i}).$$

One thus obtains the nested simulation approximation:

$$\mathrm{CVA}_0^N = \frac{1-R}{N_{\mathrm{CVA}}} \sum_{i=1}^{N_{\mathrm{CVA}}} \left(\frac{1}{N_{\mathrm{MtM}}} \sum_{j=1}^{N_{\mathrm{MtM}}} \left(\mathcal{P}^{i,j}_T - \mathcal{P}^{i,j}_{\tau_c^i}\right) + \mathcal{P}^i_{\tau_c^i} - \mathcal{P}^i_{(\tau_c^i)^-} \right)^+.$$

Similar computations can be made for the other XVAs.

The number of simulations

Assume that $\hat{\cdot}$ denotes an unbiaised estimator, we compute the mean square error MSE with the Monte-Carlo nested approximation:

$$
\begin{aligned}
\mathrm{MSE}^2 &= \mathbb{E}\left[\left(\widehat{\mathrm{CVA}}(\widehat{\mathrm{MtM}}) - \mathrm{CVA}(\mathrm{MtM})\right)^2\right] \\
&= \mathbb{E}\left[\left(\widehat{\mathrm{CVA}}(\widehat{\mathrm{MtM}}) - \mathbb{E}\left[\widehat{\mathrm{CVA}}(\widehat{\mathrm{MtM}})\right] + \mathbb{E}\left[\widehat{\mathrm{CVA}}(\widehat{\mathrm{MtM}})\right] - \mathrm{CVA}(\mathrm{MtM})\right)^2\right] \\
&= \mathbb{E}\left[\left(\widehat{\mathrm{CVA}}(\widehat{\mathrm{MtM}}) - \mathbb{E}\left[\widehat{\mathrm{CVA}}(\widehat{\mathrm{MtM}})\right]\right)^2\right] \\
&\quad + 2\mathbb{E}\left[\left(\widehat{\mathrm{CVA}}(\widehat{\mathrm{MtM}}) - \mathbb{E}\left[\widehat{\mathrm{CVA}}(\widehat{\mathrm{MtM}})\right]\right)\left(\mathbb{E}\left[\widehat{\mathrm{CVA}}(\widehat{\mathrm{MtM}})\right] - \mathrm{CVA}(\mathrm{MtM})\right)\right] \\
&\quad + \mathbb{E}\left[\left(\mathbb{E}\left[\widehat{\mathrm{CVA}}(\widehat{\mathrm{MtM}})\right] - \mathrm{CVA}(\mathrm{MtM})\right)^2\right] \\
&= \mathbb{E}\left[\left(\widehat{\mathrm{CVA}}(\widehat{\mathrm{MtM}}) - \mathbb{E}\left[\widehat{\mathrm{CVA}}(\widehat{\mathrm{MtM}})\right]\right)^2\right] \\
&\quad + 2\mathbb{E}\left[\left(\widehat{\mathrm{CVA}}(\widehat{\mathrm{MtM}}) - \mathbb{E}\left[\widehat{\mathrm{CVA}}(\widehat{\mathrm{MtM}})\right]\right)\right]\left(\mathbb{E}\left[\widehat{\mathrm{CVA}}(\widehat{\mathrm{MtM}})\right] - \mathrm{CVA}(\mathrm{MtM})\right) \\
&\quad + \left(\mathbb{E}\left[\widehat{\mathrm{CVA}}(\widehat{\mathrm{MtM}})\right] - \mathrm{CVA}(\mathrm{MtM})\right)^2 \\
&= \mathbb{E}\left[\left(\widehat{\mathrm{CVA}}(\widehat{\mathrm{MtM}}) - \mathbb{E}\left[\widehat{\mathrm{CVA}}(\widehat{\mathrm{MtM}})\right]\right)^2\right] \\
&\quad + 2\left(\mathbb{E}\left[\widehat{\mathrm{CVA}}(\widehat{\mathrm{MtM}})\right] - \mathbb{E}\left[\widehat{\mathrm{CVA}}(\widehat{\mathrm{MtM}})\right]\right)\left(\mathbb{E}\left[\widehat{\mathrm{CVA}}(\widehat{\mathrm{MtM}})\right] - \mathrm{CVA}(\mathrm{MtM})\right) \\
&\quad + \left(\mathbb{E}\left[\widehat{\mathrm{CVA}}(\widehat{\mathrm{MtM}})\right] - \mathrm{CVA}(\mathrm{MtM})\right)^2 \\
&= \mathbb{E}\left[\left(\widehat{\mathrm{CVA}}(\widehat{\mathrm{MtM}}) - \mathbb{E}\left[\widehat{\mathrm{CVA}}(\widehat{\mathrm{MtM}})\right]\right)^2\right] + \left(\mathbb{E}\left[\widehat{\mathrm{CVA}}(\widehat{\mathrm{MtM}})\right] - \mathrm{CVA}(\mathrm{MtM})\right)^2
\end{aligned}
$$

The first term is a variance-like term, of order $O(N_{\mathrm{CVA}}^{-1})$, while the second term is a bias term, which can be Taylor expanded as follows, using that $\mathbb{E}\left[\widehat{\mathrm{CVA}}(\widehat{\mathrm{MtM}})\right] = \mathbb{E}\left[\mathrm{CVA}(\widehat{\mathrm{MtM}})\right]$,

$$
\begin{aligned}
&\mathbb{E}\left[\widehat{\mathrm{CVA}}(\widehat{\mathrm{MtM}})\right] - \mathrm{CVA}(\mathrm{MtM}) \\
&= \mathbb{E}\left[\mathrm{CVA}(\widehat{\mathrm{MtM}}) - \mathrm{CVA}(\mathrm{MtM})\right] \\
&= \partial_{\mathrm{MtM}}\mathrm{CVA}(\mathrm{MtM})\mathbb{E}\left[\widehat{\mathrm{MtM}} - \mathrm{MtM}\right] + \frac{1}{2}\partial^2_{\mathrm{MtM}^2}\mathrm{CVA}(\mathrm{MtM})\mathbb{E}\left[\left(\widehat{\mathrm{MtM}} - \mathrm{MtM}\right)^2\right] + \mathbb{E}\left[O((\widehat{\mathrm{MtM}} - \mathrm{MtM})^2)\right] \\
&= \frac{1}{2}\partial^2_{\mathrm{MtM}^2}\mathrm{CVA}(\mathrm{MtM})\mathbb{E}\left[\left(\widehat{\mathrm{MtM}} - \mathrm{MtM}\right)^2\right] + \mathbb{E}\left[O((\widehat{\mathrm{MtM}} - \mathrm{MtM})^2)\right]
\end{aligned}
$$

Here, the first order term vanishes as $\widehat{\text{MtM}}$ is an unbiaised estimator of MtM. The second order term is a variance-like term of order $O(N_{\text{MtM}}^{-1})$.

In conclusion, we obtain:

$$\text{MSE}_{\text{CVA}}^2 = O(N_{\text{CVA}}^{-1}) + O(N_{\text{MtM}}^{-2}),$$

which thus suggest to take N_{MtM} of the order of $\sqrt{N_{\text{CVA}}}$, i.e. the number of inner simulations is much smaller than the number of outer simulations.

Similarly, one can prove that a n-layered nested Monte-Carlo algorithm with $M_{(0)} \otimes \cdots \otimes M_{(0)}$ simulations (with $M_{(0)}$ the number of outer simulations) is as accurate as a NMC with $M_{(0)} \otimes \sqrt{M_{(0)}} \otimes \cdots \otimes \sqrt{M_{(0)}}$ simulations.

In practice, to compute the KVA (hence the full dependence tree needs to be considered), the NMC is implemented with GPU Programming.

4.5.2 Deep Learning

An alternative to the Nested Monte-Carlo approach is find functions of time and risk-factors which approximate some intermediate XVA values at fixed time and given the risk-factors observed up to that time. Given these approximating functions, the corresponding intermediate layers drop out of the tree. One needs to *learn* an approximating function, given a sample of simulated risk-factors. Then, the approximating function is to be used for values of risk-factors which were not in the learning sample: this is the *generalization* power of the approximation.

In our XVAs context, one need to learn the XVA processes, and also some conditional VaR and conditional $\mathbb{ES}$. Thus, given a $\mathcal{F}_T$-measurable random variable X, one needs to be able to compute the following $\mathcal{F}_t$-measurable random variables

$$\begin{aligned}
&\mathbb{E}_t[X] = \mathbb{E}[X \mid \text{RF}] = \Phi(\text{RF}),\\
&\text{VaR}_{t,\alpha}(X) = \text{VaR}(X \mid \text{RF}) = \Psi(\text{RF}),\\
&\mathbb{ES}_{t,\alpha}(X) = \mathbb{ES}(X \mid \text{RF}) = \Theta(\text{RF}),
\end{aligned}$$

where $\text{RF} = (\text{RF}_1, \ldots, \text{RF}_N)$ is the $\mathcal{F}_t$-measurable random vector representing the t-value of the N underlying risk-factors. One thus needs to compute approximations for the unknown functions Φ, Ψ and Θ.

To do so, we implement a deep learning algorithm. To learn the approximating functions, we must find so-called *loss functions* for which the unknown function is a unique minimizer, and a gradient descent will allow to find a neural network whose coefficients give an approximation for our functions.

It is well-known that the value at risk is *elicitable*, meaning that the function Ψ is the solution to a minimisation problem. However, it is also well-known that the Expected Shortfall is not elicitable, i.e. not the solution of an optimization problem. This issue is solved as, in fact, the *couple* value at risk and expected shortfall (at the same level α) is jointly elicitable. Indeed, (Ψ, Θ) is the minimizer of the following function, defined over all mesurable functions,

$$(q, s) \mapsto \mathbb{E}\Big[(1-\alpha)^{-1}(f(X) - f(q(\mathrm{RF})))^+ + f(q(\mathrm{RF})) + g(s(\mathrm{RF})) \\ - g'(s(\mathrm{RF}))(s(\mathrm{RF}) - q(\mathrm{RF})) - (1-\alpha)^{-1}(X - q(\mathrm{RF}))^+\Big],$$

where f and g are functions which can be chosen as $f(z) = z$ and $g(z) = -\ln(1 + e^{-z})$ for example.

To compute the conditional expectations $\mathbb{E}[X \mid \mathrm{RF}] = \Phi(\mathrm{RF})$, one can use the mean square error: Φ minimises

$$h \mapsto \mathbb{E}\left[(h(\mathrm{RF}) - X)^2\right]$$

Note that it is equivalent to minimizing the square distance to the conditional expectation (which we can compute using NMC as before)

$$h \mapsto \mathbb{E}\left[(h(\mathrm{RF}) - \mathbb{E}[X \mid \mathrm{RF}])^2\right].$$

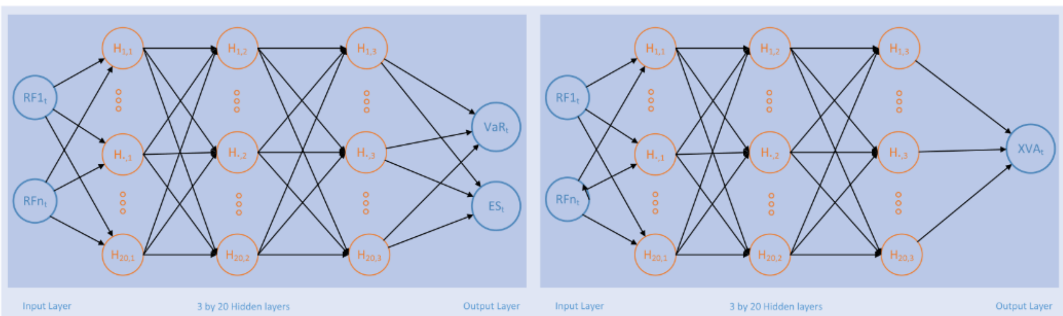
Indeed,

$$\begin{aligned} &\mathbb{E}\left[(h(\mathrm{RF}) - X)^2\right] \\ &= \mathbb{E}\left[(h(\mathrm{RF}) - \mathbb{E}[X \mid \mathrm{RF}])^2\right] \\ &\quad + \mathbb{E}\left[(\mathbb{E}[X \mid \mathrm{RF}] - X)^2\right] \\ &\quad + 2\mathbb{E}\left[(h(\mathrm{RF}) - \mathbb{E}[X \mid \mathrm{RF}])(\mathbb{E}[X \mid \mathrm{RF}] - X)\right] \\ &= \mathbb{E}\left[(h(\mathrm{RF}) - \mathbb{E}[X \mid \mathrm{RF}])^2\right] + \mathbb{E}[\mathrm{VaR}(X \mid \mathrm{RF})] \end{aligned}$$

as the term $2\mathbb{E}\left[(h(\mathrm{RF}) - \mathbb{E}[X \mid \mathrm{RF}])(\mathbb{E}[X \mid \mathrm{RF}] - X)\right]$ vanishes by definition of conditional expectation.

1. Simulate a M-sample of the risk-factors $\mathrm{RF}^i = (\mathrm{RF}^i_1, \ldots, \mathrm{RF}^i_N)$, $1 \le i \le M$.
2. For each $1 \le i \le M$,

The approximating functions Φ, Ψ, Θ are looked for among neural networks, which have the following structure



	CVA	FVA	IM	MVA	Gap CVA†	EC	KVA
Hidden layers	3	5	3	3	3	3	3
Hidden layer size	20	6	20	20	20	20	20
Learning rate	0.025	0.025	0.05	0.1	0.1	0.025	0.1
Momentum	0.95	0.95	0.5	0.5	0.5	0.95	0.5
Iterations	100	50	150	100	100	100	100
Loss function	MSE	MSE	(44)	MSE	(44)	(44)	MSE
Application	Netting set	Portf.	Netting set	Netting set	Netting set	Portf.	Portf.

For the left-hand network, the features RF are the state variables (risk-factors) and the labels are the pathwise XVA items for which we want to compute the conditional Var and $\mathbb{ES}$, for example the loss function with 1 year increment. The output is the joint estimate of Var and $\mathbb{ES}$ of the label given the features, at selected level α.
For the right-hand network, the features are still the risk-factors RF, and the labels the pathwise XVA items which express as comditional expectation of a functional of the risk-factors. The output is the pathwise conditional mean of the label given the factors.

Then the Deep XVAs algorithm is as follows

Algorithm 1 Deep XVAs algorithm.

- Simulate forward m realizations (Euler paths) of the market risk factor processes and of the counterparty survival indicator processes (i.e. default times) on a refined time grid;
- For each pricing time $t = t_i$ of a pricing time grid, with coarser time step denoted by h, and for each counterparty c:
  - Learn the corresponding $\mathbb{VaR}_t$ and $\mathbb{ES}_t$ terms visible in (A11) or (under the time-discretized outer integral in) (A13);
  - Learn the corresponding $\mathbb{E}_t$ terms visible in (A12) through (A14);
  - Compute the ensuing pathwise CVA and MVA as per (A12)–(A14);
- For $\mathrm{FVA}^{(0)}$, consider the following time discretization of (A9) (in which λ is the risky funding spread process of the bank) with time step h:

$$\mathrm{FVA}_t^{(0)} \approx \mathbb{E}_t[\mathrm{FVA}_{t+h}^{(0)}] + h\lambda_t \left(\sum_c J_t^c (P_t^c - \mathrm{VM}_t^c) - \mathrm{CVA}_t - \mathrm{MVA}_t - \mathrm{FVA}_t^{(0)} \right)^+ \quad (1)$$

  and, for each $t = t_i$, learn the corresponding $\mathbb{E}_t$ in (46), then solve the semi-linear equation for $\mathrm{FVA}_t^{(0)}$;
- For each Picard iteration k (until numerical convergence), simulate forward $L^{(k)}$ as per the first line in (A10) (which only uses known or already learned quantities), and:
  - For economic capital $\mathrm{EC}^{(k)}$, for each $t = t_i$, learn $\mathbb{ES}_t\big((L^{(k)})^\circ_{t+1} - (L^{(k)})^\circ_t\big)$ (cf. Definition A.1);
  - $\mathrm{KVA}^{(k)}$ and $\mathrm{FVA}^{(k)}$ then require a backward recursion solved by deep learning approximation much like the one for $\mathrm{FVA}^{(0)}$ above.

4.5.3 Case study

Risk factors : 10 interest rates following a one factor Hull&White model, 9 exchange rates following a Black&Scholes model, and 11 Cox-Ingersoll-Ross default intensity processes. The default times are jointly modelled by a common shock model as described above. This setup results in about 40 risk factors used as the deep learning features.

We consider a bank portfolio of 10K randomly generated swap trades with

- trade currency and counterparty uniform on $[1, 10]$,
- Notional uniform on $[10K, 100K]$,

- Some trades are collateralized with VM = MtM, PIM is the 99% gap risk value at risk and RIM is the 75% gap risk value at risk,

- Economic capital is the 97.5% expected shortfall of 1-year ahead trading loss of the bank shareholders.

Numerical tests allow to quantify the impact on the XVAs of the collateralization, and of the inclusion of the capital at risk in the computation of the FVA. Also, one observes that the fact that KVA is loss-absorbing has an impact on the KVA.

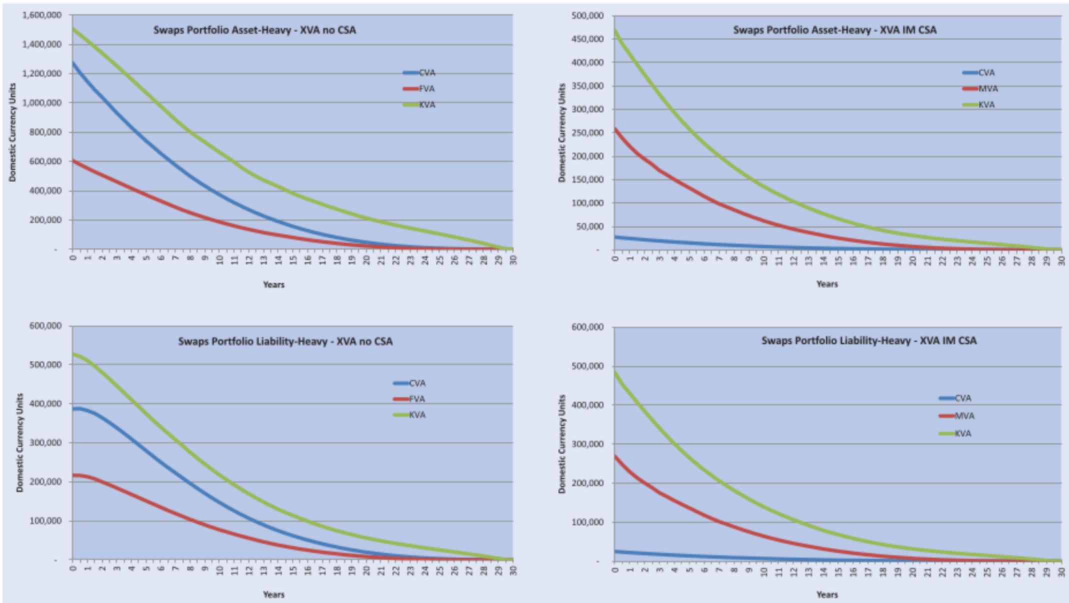


Figure 12. *(Top left)* Asset-heavy portfolio, no CSA. *(Top right)* Asset-heavy portfolio under CSA. *(Bottom left)* Liability–heavy portfolio, no CSA. *(Bottom right)* Liability-heavy portfolio under CSA.

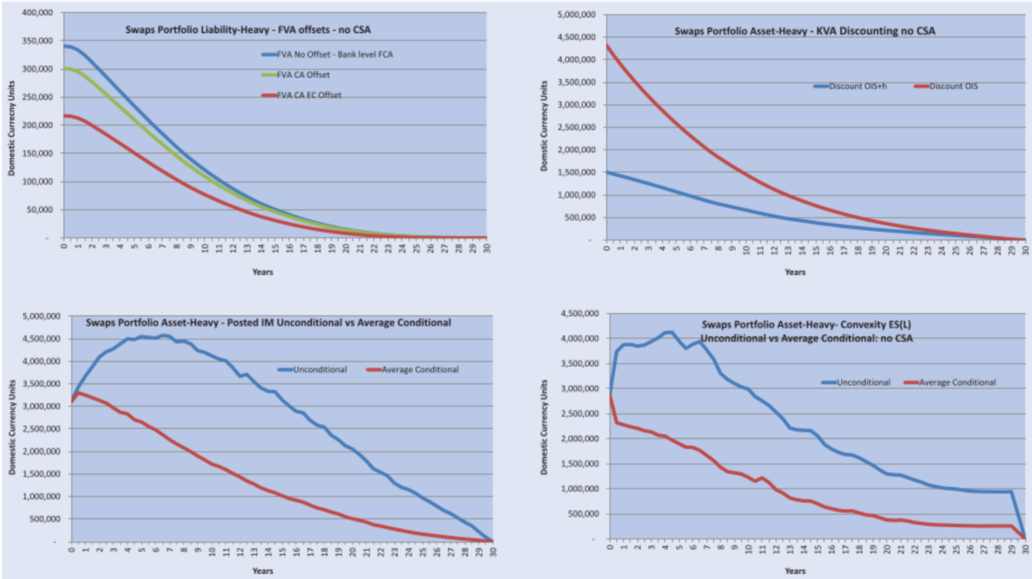


Figure 13. *(Top left)* FVA ignoring the off-setting impact of reserve capital and capital at risk, cf. section 3.4 (blue), FVA as per (A9) accounting for the off-setting impact of reserve capital but ignoring the one of capital at risk (green), refined FVA as per (A4) accounting for both impacts (red). *(Top right)* KVA ignoring the off-setting impact of the risk margin, i.e. with CR instead of (CR − KVA) in (A8) (red), refined KVA as per (A6)–(A7) (blue). *(Bottom left)* In the case of the asset-heavy portfolio under CSA, unconditional PIM profile, i.e. with $\mathbb{V}\mathrm{a}\mathbb{R}_t$ replaced by $\mathbb{V}\mathrm{a}\mathbb{R}$ in (A11) (blue), vs. pathwise PIM profile, i.e. mean of the pathwise PIM process as per (A11) (red). *(Bottom right)* In the asset-heavy portfolio no CSA case, unconditional economic capital profile, i.e. EC profile ignoring the words 'time-t conditional' in Definition A.1 (blue), vs. pathwise economic capital profile, i.e. mean of the pathwise EC process as per Definition A.1 (red).