TSP Dantzig-Wolfe Decomposition - Column generation with 1-trees

Correction

X is the set of 1-Trees (contained in the complete graph). The complete graph K_n has *n* vertices numbered 1,…,*n*. Edges of K_n are denoted by *E*. Each edge *e* of the complete graph has a cost c_e.

A 1-tree is a partial graph of K_n such that vertices 2, 3,…, *n* are covered by a tree and vertex 1 is connected by 2 edges to two vertices in 2, 3,…, *n*. The degree of vertex 1 is equal to 2. See the example below in K_5.

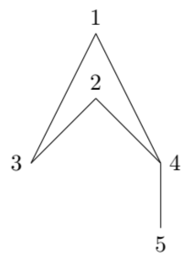

Figure 1: exemple de 1-arbre à 5 sommets

<u>Question 1.</u>

We search for a 1-Tree of degree 2 on every vertices, and of minimal cost.

A 1-tree with degree 2 on each vertex is a cycle that goes through each vertex. So, we search an hamiltonian cycle of minimal cost.

We can modelize this problem by the following program (M) in 0-1 variables :

$$\min_{\lambda} \sum_{i=1}^{|X|} c(\chi_i)\lambda_i$$

Subject to
$$\begin{cases} \sum_{i=1}^{|X|} d_j(\chi_i)\lambda_i = 2 \quad j = 2,\dots,n \text{ (constaints degree for vertex j)} \\ \sum_{i=1}^{|X|} \lambda_i = 1 \qquad (convexity) \\ \lambda_i \in \{0,1\}\ i = 1,\dots,|X| \end{cases}$$

with

$c(\chi)$ cost of 1-Tree χ

$d_j(\chi)$ degree of vertex j in 1-Tree χ

The convexity constraint is to select exactly one 1-tree

Question 2.

Now, we consider linear program (ML) where the 0-1 variables λ_i are relaxed to $\lambda_i \geq 0$.

μ_j is the dual variable related to the constraint degree of vertex j=2,…,n and η dual variable related to convexity constraint.

Now, we concentre on the subproblem. The aim of the subproblem is to find a variable of minimum reduced cost.

In order to simplify the presentation, the best is to introduce virtually a constraint degree on vertex 1 with a dual variable $\mu_1 = 0$.

In order to describe 1-tree χ_i let us introduce the following notations :

$a_{i,e} = 1\ if\ edge\ e\ is\ in\ 1 - tree\ \chi_i$ and $a_{i,e} = 0\ otherwise$

The cost of variable λ_i is $c(\chi_i) = \sum_{e \in E} c_e a_{i,e}$

$\delta(j)$ is the set of edges *e* starting from j : $\delta(j) = \{e \in E : j \in e\}$

The degree of vertex j in 1-tree χ_i is $d_j(\chi_i) = \sum_{e \in \delta(j)} a_{i,e}$

Reduced cost of λ_i is $c(\chi_i) - \sum_{j=1}^{n} \mu_j d_j(\chi_i) \quad - \eta = \sum_{e \in E} c_e a_{i,e} - \sum_{j=1}^{n} \mu_j \sum_{e \in \delta(j)} a_{i,e} - \eta =$
$\sum_{e=(j,j') \in E} \left(c_e - \mu_j - \mu_{j'}\right) a_{i,e} - \eta$

So solving the subproblem is equivalent to search a 1-tree of minimum cost with edge cost $c'_{e=(j,j')} = c_e - \mu_j - \mu_{j'}$. This is done by solving minimum cost tree (with cost c') on vertices 2, 3,…, *n* and once this is done, by adding the 2 minimum cost edges (with cost c') starting from vertex 1. The minimum cost tree can be solved by polynomial time Kruskal algorithm.

Question 3

We consider K_5. The edge costs are the following :

Costs	Vertex 1	Vertex 2	Vertex 3	Vertex 4	Vertex 5
Vertex 1	.	7	2	1	5
Vertex 2		.	3	6	8
Vertex 3			.	4	2
Vertex 4				.	9

We consider the following 1-trees

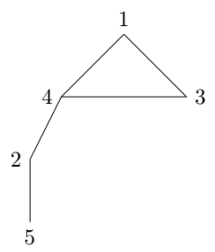

Figure 2: 1-arbre numéro 1

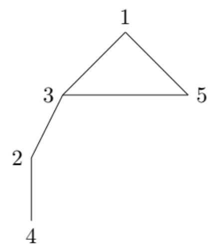

Figure 3: 1-arbre numéro 2

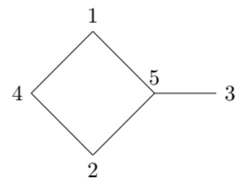

Figure 4: 1-arbre numéro 3

<u>Question 3.1</u>

We write the problem (ML) restricted to the three 1-trees given above.

(MLR)

$$\min_{\lambda} 21\lambda_1 + 18\lambda_2 + 22\lambda_3$$

$$\text{Subject to} \begin{cases} 2\lambda_1 + 2\lambda_2 + 2\lambda_3 = 2 \quad (v2) \\ 2\lambda_1 + 3\lambda_2 + 1\lambda_3 = 2 \quad (v3) \\ 3\lambda_1 + 1\lambda_2 + 2\lambda_3 = 2 \quad (v4) \\ 1\lambda_1 + 2\lambda_2 + 3\lambda_3 = 2 \quad (v5) \\ \\ \lambda_1 + \lambda_2 + \lambda_3 = 1 \quad (convexity) \\ \lambda_1, \lambda_2, \lambda_3 \geq 0 \end{cases}$$

<u>Question 3.2</u>

Here, we give the primal and dual solutions of the previous (MLR), and we want to find a new column if there is, to introduce in (MLR).

Primal solution $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{3}$ Primal objective=20+1/3

Dual solution $\mu_2 = 2, \mu_3 = 0, \mu_4 = 2 + \frac{1}{3}, \mu_5 = 2 - \frac{1}{3}\ \ , \eta = 8 + \frac{1}{3}$

Dual Objective=12+8+1/3=20+1/3

We can check the dual solution : reduced costs of $\lambda_1, \lambda_2, \lambda_3$ must be nonnegative.

For λ_1 : $21 - 2\mu_2 - 2\mu_3 - 3\mu_4 - \mu_5 - \eta$ = 21- 4 -0 -6-1 –2+1/3 - 8-1/3=21 – 21 = 0

For λ_2 : $18 - 2\mu_2 - 3\mu_3 - \mu_4 - 2\mu_5 - \eta$ = 18 - 4 -0 -2 -1/3 –4+2/3 - 8-1/3=18 – 18 = 0

For λ_3 : $22 - 2\mu_2 - \mu_3 - 2\mu_4 - 3\mu_5 - \eta$ = 22 - 4 -0 -4 -2/3 –6+1 - 8-1/3 = 22 – 22 = 0

So, the primal and dual solutions are optimal.

Solving subproblem.

Data : costs and dual variables

µ	Vertex1	Vertex2 = 2	Vertex3 = 0	Vertex4=2+1/3	Vert.5=2-1/3
Vertex1	.	7	2	1	5
Vertex2 = 2		.	3	6	8
Vertxx3 = 0			.	4	2
Vertex4=2+1/3				.	9

Then reduced costs

µ	Vertex1	Vertex2 = 2	Vertex3 = 0	Vertex4=2+1/3	Vert.5=2-1/3
Vertex1	.	7-2	2	1-2-1/3	5-2+1/3
Vertex2 = 2		.	3-2	6-2-2-1/3	8-2-2+1/3
Vertex3 = 0			.	4-2-1/3	2-2+1/3
Vertex4=2+1/3				.	9-2-1/3-2+1/3

µ	Vertex1	Vertex2 = 2	Vertex3 = 0	Vertex4=2+1/3	Vert.5=2-1/3
Vertex1	.	5	2	-1-1/3	3+1/3
Vertex2 = 2		.	1	2-1/3	4+1/3
Vertex3 = 0			.	2-1/3	1/3
Vertex4=2+1/3				.	5

We compute a minimum spanning tree on vertices 2, 3 ,4, 5

Edge (3,5) reduced cost 1/3,

Edge (2,3) reduced cost 1

Edege (2,4) reduced cost 2-1/3

Then we add the two edges of minimum reduced costs starting from vertex 1

Edges (1,4) reduced cost -1-1/3

Edge (1,3) reduced cost 2,

Reduced cost of this 1-tree is : 4-1/3 plus -8-1/3 (for the convexity constraint) which is $-4-\frac{2}{3}<0$

The cost of this 1-tree is 14. Degrees for vertices from 2 to 5 are 2, 3, 2, 1. We can check the reduced cost of this 1-tree as if this column were in (MLR) :

reduced cost is $14-2\mu_2-3\mu_3-2\mu_4-1\mu_5-\eta=14-4-0-4-\frac{2}{3}-2+\frac{1}{3}-8-\frac{1}{3}=4-\frac{1}{3}-8-\frac{1}{3}=-4-\frac{2}{3}$. The result is correct.

A lower bound of (ML) is the value of (MLR) plus the reduced cost = 20+1/3 – 4 -2/3 =16 -1/3.

We denote by λ_4 the variable of the new 1-tree and add this variable to (MLR)

Question 3.3

The new (MLR) is the following :

(MLR)

$$\min_{\lambda} 21\lambda_1 + 18\lambda_2 + 22\lambda_3 + 14\lambda_4$$

$$\text{Subject to } \begin{cases} 2\lambda_1 + 2\lambda_2 + 2\lambda_3 + 2\lambda_4 = 2 \quad (v2) \\ 2\lambda_1 + 3\lambda_2 + 1\lambda_3 + 3\lambda_4 = 2 \quad (v3) \\ 3\lambda_1 + 1\lambda_2 + 2\lambda_3 + 2\lambda_4 = 2 \quad (v4) \\ 1\lambda_1 + 2\lambda_2 + 3\lambda_3 + 1\lambda_4 = 2 \quad (v5) \\ \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1 \quad (convexity) \\ \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0 \end{cases}$$

Here, we give the primal and dual solutions of this last (MLR), and we want to find a new column if there is, to introduce in (MLR).

Primal solution $\lambda_1 = \lambda_2 = 0, \lambda_3 = \lambda_4 = \frac{1}{2}$; Primal objective=18

Dual solution $\mu_2 = 3, \mu_3 = 0, \mu_4 = 0$, $\mu_5 = 4$, $\eta = 4$; Dual objective=14+4=18

We can check the dual solution i.e. reduced costs of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ must be nonnegative.

For λ_1 : $21 - 2\mu_2 - 2\mu_3 - 3\mu_4 - \mu_5 - \eta$ = 21- 6 -0 -0 -4 -4=21 – 14 ≥ 0

For λ_2 : $18 - 2\mu_2 - 3\mu_3 - \mu_4 - 2\mu_5 - \eta$ = 18 - 6 -0 -0 -8 -4=18 – 18 = 0

For λ_3 : $22 - 2\mu_2 - \mu_3 - 2\mu_4 - 3\mu_5 - \eta$ = 22 - 6 -0 -0 -12 -4 = 22 – 22 = 0

For λ_4 : $14 - 2\mu_2 - 3\mu_3 - 2\mu_4 - \mu_5 - \eta$ = 14 - 6 -0 -0 -4 -4 = 14 – 14 = 0

So, the primal and dual solutions are optimal.

Solving the subproblem

Data : costs and dual variables

μ	Vertex1	Vertex2 = 3	Vertex3 = 0	Vertex4 = 0	Vertex5 = 4
Vertex1	.	7	2	1	5
Vertex2 = 3		.	3	6	8
Vertxx3 = 0			.	4	2
Vertex4 = 0				.	9

Reduced costs

μ	Vertex1	Vertex2 = 3	Vertex3 = 0	Vertex4 = 0	Vertex5 = 4
Vertex1	.	7-3	2	1	5-4
Vertex2 = 3		.	3-3	6-3	8-3-4
Vertxx3 = 0			.	4	2-4
Vertex4 = 0				.	9-4

μ	Vertex1	Vertex2 = 3	Vertex3 = 0	Vertex4 = 0	Vertex5 = 4
Vertex1	.	4	2	1	1
Vertex2 = 3		.	0	3	1
Vertxx3 = 0			.	4	-2
Vertex4 = 0				.	5

We compute a minimum spanning tree on vertices 2, 3, 4, 5

Edge (3,5) reduded cost= -2

Edge (2,3) reduced cost=0

→ Edge (2,5) reduced cost=1, is not introduced because it makes a cycle

Edge (2,4) reduced cost=3

Then we add the two edges of minimum reduced costs starting from vertex 1

Edge (1,4) reduced cost=1

Edge (1,5) reduced cost=1

The reduced cost of this 1-tree is : 3 plus -4 (for the convexity constraint) . The total is -1

We note that this 1-tree is a cycle because the degree of each vertex is equal to 2. The cost of this 1-tree is 17.

A lower bound of (ML) is the value of (MLR) plus the reduced cost = 18 – 1 =17.

So, the cycle is reaching this lower bound, so it is optimal. We can stop the algorithm.