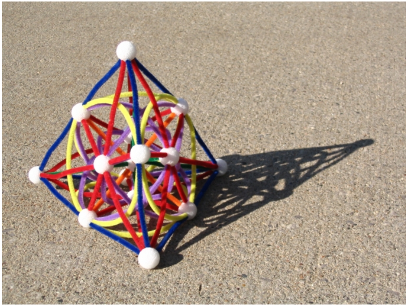
Building some Finite Models of Projective Space Geometry in Coq



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groupe LTP (GDR GPL) - 6 dec. 2018

Outline

1. Motivations and Context
2. Examples pg(3,2) and pg(3,3)
3. Coq specifications
4. Proof Optimizations
5. Results and Future Work

Projective Space Geometry

- Incidence Geometry
  - points, lines and an incidence relation
- Projective Incidence Geometry
  - in 2D : 2 lines always intersect
  - in 3D : Pasch's axiom
- Simple description : only 6 axioms
- Finite Models in Coq
  - focusing on 3D models : pg(3,2), . . .
  - taking Coq to its limits (w.r.t. specification and w.r.t. proof)

Objects and Operations

- Objects : Point, Line

```
Parameter Point, Line : Type.
```

- Incidence relation : incid_lp

```
Parameter incid_lp : Point -> Line -> bool.
```

- Boolean equalities on points and lines : eqP, eqL

```
Parameter eqP : Point -> Point -> bool.
Parameter eqL : Line -> Line -> bool.
```

- All distinct for points and lines : dist_3p, dist_4p, dist_3l

```
Definition dist_3p (A B C :Point) : bool :=
(negb (eqP A B)) && (negb (eqP A C)) && (negb (eqP B C)).

Definition dist_4p (A B C D:Point) : bool := ...
Definition dist_3l (A B C :Line) : bool := ...
```

- Intersection of 2 lines : Intersect_In

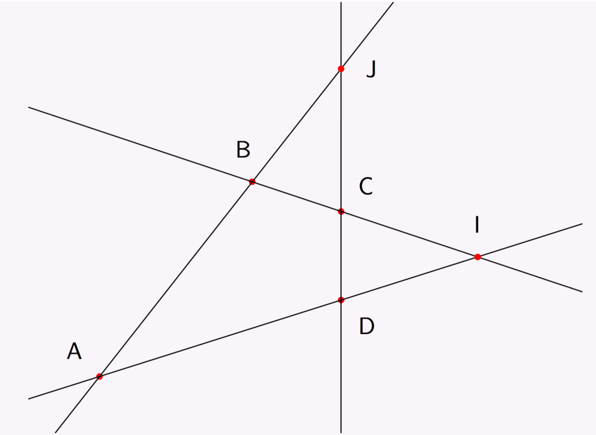
```
Definition Intersect_In (l1 l2 :Line) (P:Point) :=
incid_lp P l1 && incid_lp P l2.
```

Axioms for Projective Space Geometry : from a geometry point of view

- a1 : throught 2 points, there is one line.
- uniqueness : Given 2 points and 2 lines, if the 2 points are both on both lines, either the points are equal, or the lines.
- a2 : Pasch's axiom (if 2 lines intersect. . .).
- a3_1 : Each line has at least 3 points.
- a3_2 : There exists 2 lines which do not intersect (dim>2).
- a3_3 : Given 3 distinct lines, there exists a fourth one which intersects with all three (dim<=3).

Axioms for Projective Space Geometry : from a geometry point of view

- Pasch's axiom



Axioms for Projective Space Geometry : from a geometry point of view

```
Axiom a1_exists : forall A B : Point, { l : Line| incid_lp A l && incid_lp B l}.

Axiom uniqueness : forall (A B :Point)(l1 l2:Line),
incid_lp A l1 -> incid_lp B l1 -> incid_lp A l2 -> incid_lp B l2 -> A = B \/ l1 = l2.

Axiom a2 : forall A B C D:Point, forall lAB lCD lAC lBD :Line, dist_4p A B C D ->
incid_lp A lAB && incid_lp B lAB -> incid_lp C lCD && incid_lp D lCD ->
incid_lp A lAC && incid_lp C lAC -> incid_lp B lBD && incid_lp D lBD ->
(exists I:Point, incid_lp I lAB && incid_lp I lCD) ->
exists J:Point, incid_lp J lAC && incid_lp J lBD.

Axiom a3_1 : forall l:Line,
{A:Point & {B:Point & {C:Point |
(dist_3p A B C) && (incid_lp A l && incid_lp B l && incid_lp C l)}}}.

Axiom a3_2 : exists l1:Line, exists l2:Line,
forall p:Point,  (incid_lp p l1 && incid_lp p l2).

Axiom a3_3 : forall l1 l2 l3:Line, dist_3l l1 l2 l3 ->
exists l4 :Line, exists J1:Point, exists J2:Point, exists J3:Point,
Intersect_In l1 l4 J1 && Intersect_In l2 l4 J2 && Intersect_In l3 l4 J3.
```

Axioms for Projective Space Geometry : from a logic point of view

```
Axiom a1_exists : forall A B : Point, { l : Line| incid_lp A l && incid_lp B l}.

Axiom uniqueness : forall (A B :Point)(l1 l2:Line),
incid_lp A l1 -> incid_lp B l1 -> incid_lp A l2 -> incid_lp B l2 -> A = B \/ l1 = l2.

Axiom a2 : forall A B C D:Point, forall lAB lCD lAC lBD :Line, dist_4p A B C D ->
incid_lp A lAB && incid_lp B lAB -> incid_lp C lCD && incid_lp D lCD ->
incid_lp A lAC && incid_lp C lAC -> incid_lp B lBD && incid_lp D lBD ->
(exists I:Point, incid_lp I lAB && incid_lp I lCD) ->
exists J:Point, incid_lp J lAC && incid_lp J lBD.

Axiom a3_1 : forall l:Line,
{A:Point & {B:Point & {C:Point |
(dist_3p A B C) && (incid_lp A l && incid_lp B l && incid_lp C l)}}}.

Axiom a3_2 : exists l1:Line, exists l2:Line,
forall p:Point,  (incid_lp p l1 && incid_lp p l2).

Axiom a3_3 : forall l1 l2 l3:Line, dist_3l l1 l2 l3 ->
exists l4 :Line, exists J1:Point, exists J2:Point, exists J3:Point,
Intersect_In l1 l4 J1 && Intersect_In l2 l4 J2 && Intersect_In l3 l4 J3.
```

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Examples : pg(3,q)

	# points	# lines	# points per line
$pg(3,2)$	15	35	3
$pg(3,3)$	40	130	4
$pg(3,4)$	85	357	5
$pg(3,q)$	$(q^2+1)(q+1)$	$(q^2+q+1)(q^2+1)$	$q+1$

- By duality : # planes = # points.
- Describing the incidence relation of pg(3, q) :
  for each line, we provide the q+1 points which belong to it.
- e.g. pg(3,3) [1]

1. Alan R. Prince. Projective planes of order 12 and PG(3,3). Discrete Mathematics, 208-209 :477-483, 1999.

pg(3,3) - description of the incidence relation

*L*1	0	1	4	13	*L*14	1	2	5	14	*L*27	1	7	12	33	*L*40	1	8	24	26	*L*53	1	10	37	38
*L*2	0	2	24	17	*L*15	2	3	6	15	*L*28	2	19	26	4	*L*41	2	22	12	32	*L*54	2	33	29	13
*L*3	0	3	12	39	*L*16	3	5	27	20	*L*29	3	38	32	24	*L*42	3	10	26	28	*L*55	3	4	7	16
*L*4	0	5	26	34	*L*17	5	6	9	18	*L*30	5	30	28	12	*L*43	5	33	32	36	*L*56	5	24	19	13
*L*5	0	6	32	11	*L*18	6	27	35	1	*L*31	6	16	36	26	*L*44	6	4	28	21	*L*57	6	12	38	17
*L*6	0	27	28	31	*L*19	27	9	25	2	*L*32	27	13	21	32	*L*45	27	24	36	23	*L*58	27	26	30	39
*L*7	0	9	36	37	*L*20	9	35	14	3	*L*33	9	17	23	28	*L*46	9	12	21	8	*L*59	9	32	16	34
*L*8	0	35	21	29	*L*21	35	25	15	5	*L*34	35	39	8	36	*L*47	35	26	23	22	*L*60	35	28	13	11
*L*9	0	25	23	7	*L*22	25	14	20	6	*L*35	25	34	22	21	*L*48	25	32	8	10	*L*61	25	36	17	31
*L*10	0	14	8	19	*L*23	14	15	20	6	*L*36	14	11	10	23	*L*49	14	28	22	33	*L*62	14	21	39	37
*L*11	0	15	22	38	*L*24	15	20	1	9	*L*37	15	31	33	8	*L*50	15	36	10	4	*L*63	15	23	34	29
*L*12	0	20	10	30	*L*25	20	18	2	35	*L*38	20	37	4	22	*L*51	20	21	33	24	*L*64	20	8	11	7
*L*13	0	18	33	16	*L*26	18	1	3	25	*L*39	18	29	24	10	*L*52	18	23	4	12	*L*65	18	22	31	19

*L*66	1	11	21	31	*L*79	1	16	23	39	*L*92	1	17	19	34	*L*105	1	22	30	36	*L*118	1	28	29	32
*L*67	2	31	23	37	*L*80	2	13	8	34	*L*93	2	39	38	11	*L*106	2	10	16	21	*L*119	2	36	7	28
*L*68	3	37	8	29	*L*81	3	17	22	11	*L*94	3	34	30	31	*L*107	3	33	13	23	*L*120	3	21	19	36
*L*69	5	29	22	7	*L*82	5	39	10	31	*L*95	5	11	16	37	*L*108	5	4	17	8	*L*121	5	23	38	21
*L*70	6	7	10	19	*L*83	6	34	33	37	*L*96	6	31	13	29	*L*109	6	24	39	22	*L*122	6	8	30	23
*L*71	27	19	33	38	*L*84	27	11	4	29	*L*97	27	37	17	7	*L*110	27	12	34	10	*L*123	27	22	16	8
*L*72	9	38	4	30	*L*85	9	31	24	7	*L*98	9	29	39	19	*L*111	9	26	11	33	*L*124	9	10	13	22
*L*73	35	30	24	16	*L*86	35	37	12	19	*L*99	35	7	34	38	*L*112	35	32	31	4	*L*125	35	33	17	10
*L*74	25	16	12	13	*L*87	25	29	26	38	*L*100	25	19	11	30	*L*113	25	28	37	24	*L*126	25	4	39	33
*L*75	14	13	26	17	*L*88	14	7	32	30	*L*101	14	38	31	16	*L*114	14	36	29	12	*L*127	14	24	34	4
*L*76	15	17	32	39	*L*89	15	19	28	16	*L*102	15	30	37	13	*L*115	15	21	7	26	*L*128	15	12	11	24
*L*77	20	39	28	34	*L*90	20	38	36	13	*L*103	20	16	29	17	*L*116	20	23	19	32	*L*129	20	26	31	12
*L*78	18	34	36	11	*L*91	18	30	21	17	*L*104	18	13	7	39	*L*117	18	8	38	28	*L*130	18	32	37	26

pg(3,3) - comments on the description

- The formal proof for the axioms fails. Why ?
- Checking the formal statement is correct.
- Checking the proof method works properly
  (yes, it works for pg(3,2)).
- Checking the description is correct.
  The incidence relation is incorrect.
- Fixing the incidence relation : using the number of points per line property, to locate the errors.

pg(3,3) - description of the incidence relation

*L*1	0	1	4	13	*L*14	1	2	5	14	*L*27	1	7	12	33	*L*40	1	8	24	26	*L*53	1	10	37	38
*L*2	0	2	24	17	*L*15	2	3	6	15	*L*28	2	19	26	4	*L*41	2	22	12	32	*L*54	2	33	29	13
*L*3	0	3	12	39	*L*16	3	5	27	20	*L*29	3	38	32	24	*L*42	3	10	26	28	*L*55	3	4	7	16
*L*4	0	5	26	34	*L*17	5	6	9	18	*L*30	5	30	28	12	*L*43	5	33	32	36	*L*56	5	24	19	13
*L*5	0	6	32	11	*L*18	6	27	35	1	*L*31	6	16	36	26	*L*44	6	4	28	21	*L*57	6	12	38	17
*L*6	0	27	28	31	*L*19	27	9	25	2	*L*32	27	13	21	32	*L*45	27	24	36	23	*L*58	27	26	30	39
*L*7	0	9	36	37	*L*20	9	35	14	3	*L*33	9	17	23	28	*L*46	9	12	21	8	*L*59	9	32	16	34
*L*8	0	35	21	29	*L*21	35	25	15	5	*L*34	35	39	8	36	*L*47	35	26	23	22	*L*60	35	28	13	11
*L*9	0	25	23	7	*L*22	25	14	20	6	*L*35	25	34	22	21	*L*48	25	32	8	10	*L*61	25	36	17	31
*L*10	0	14	8	19	*L*23	14	15	20	6	*L*36	14	11	10	23	*L*49	14	28	22	33	*L*62	14	21	39	37
*L*11	0	15	22	38	*L*24	15	20	1	9	*L*37	15	31	33	8	*L*50	15	36	10	4	*L*63	15	23	34	29
*L*12	0	20	10	30	*L*25	20	18	2	35	*L*38	20	37	4	22	*L*51	20	21	33	24	*L*64	20	8	11	7
*L*13	0	18	33	16	*L*26	18	1	3	25	*L*39	18	29	24	10	*L*52	18	23	4	12	*L*65	18	22	31	19

18 27

30

*L*66	1	11	21	31	*L*79	1	16	23	39	*L*92	1	17	19	34	*L*105	1	22	30	36	*L*118	1	28	29	32
*L*67	2	31	23	37	*L*80	2	13	8	34	*L*93	2	39	38	11	*L*106	2	10	16	21	*L*119	2	36	7	28
*L*68	3	37	8	29	*L*81	3	17	22	11	*L*94	3	34	30	31	*L*107	3	33	13	23	*L*120	3	21	19	36
*L*69	5	29	22	7	*L*82	5	39	10	31	*L*95	5	11	16	37	*L*108	5	4	17	8	*L*121	5	23	38	21
*L*70	6	7	10	19	*L*83	6	34	33	37	*L*96	6	31	13	29	*L*109	6	24	39	22	*L*122	6	8	30	23
*L*71	27	19	33	38	*L*84	27	11	4	29	*L*97	27	37	17	7	*L*110	27	12	34	10	*L*123	27	22	16	8
*L*72	9	38	4	30	*L*85	9	31	24	7	*L*98	9	29	39	19	*L*111	9	26	11	33	*L*124	9	10	13	22
*L*73	35	30	24	16	*L*86	35	37	12	19	*L*99	35	7	34	38	*L*112	35	32	31	4	*L*125	35	33	17	10
*L*74	25	16	12	13	*L*87	25	29	26	38	*L*100	25	19	11	30	*L*113	25	28	37	24	*L*126	25	4	39	33
*L*75	14	13	26	17	*L*88	14	7	32	30	*L*101	14	38	31	16	*L*114	14	36	29	12	*L*127	14	24	34	4
*L*76	15	17	32	39	*L*89	15	19	28	16	*L*102	15	30	37	13	*L*115	15	21	7	26	*L*128	15	12	11	24
*L*77	20	39	28	34	*L*90	20	38	36	13	*L*103	20	16	29	17	*L*116	20	23	19	32	*L*129	20	26	31	12
*L*78	18	34	36	11	*L*91	18	30	21	17	*L*104	18	13	7	39	*L*117	18	8	38	28	*L*130	18	32	37	26

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Coq specifications

- Point and Line as simple inductive types.
  - Case analysis is easy.
  - Finding a witness can be challenging.
    = trying each possible value and running the tactics.
  - Writing the specification is a bit boring.

    ```
    Inductive Point := P0 | P1 | P2 | ... | P40.
    ```

- Solutions
  - Using finite types (ssreflect/mathcomp)
  - Using plain inductive data-types and an external program to generate the specification

Our choice : an external program

- We choose to have an external program generating the specification (actually outputs a gallina specification)
- Indeed, we need a specification generation process anyway (for witnesses).
- Our implementation
  - plain data-types combined with boolean reflection
  - generating data-types such as Line (130 constructors)
  - incidence relation as a boolean predicate
  - equality (decidable)
  - order relation (decidable and total)
  - The witness for existential quantification are computed beforehand.

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Proof Optimizations

- Witness finding reduced to function computation

```
Definition f_a3_3 (l1:Line) (l2:Line) (l3:Line) := ...
```

  computes a line which intersects the 3 lines l1, l2 and l3.

- Factorizing proofs as lemmas.

$$\forall T\ Z\ x, \texttt{incid\_lp}\ T\ x\ \rightarrow\ \texttt{incid\_lp}\ Z\ x \rightarrow T <> Z\ \rightarrow\ x = (\texttt{l\_from\_points}\ T\ Z)$$

- Proof-engineering : sequences of tactics, abstract, par

```
par :abstract (time (case v2; intros hp1p2; first [exact (degen_bool _ hp1p2) | (case
v3; intros hp1p3 hdist x; solve [ (exact (degen_bool _ hp1p3)) | (exact (degen_bool
_ hdist)) | exists_lppp (fst x) (fst (fst (snd x))) (snd (fst (snd x))) (snd (snd
x)) ])])).
```

  no try, each goal is solved the first time it is encountered.

Symmetries

- Using appropriate symmetries to reduce the number of cases to check.

```
Axiom a2 : forall A B C D:Point, forall lAB lCD lAC lBD :Line, dist_4p A B C D ->
incid_lp A lAB && incid_lp B lAB -> incid_lp C lCD && incid_lp D lCD ->
incid_lp A lAC && incid_lp C lAC -> incid_lp B lBD && incid_lp D lBD ->
(exists I:Point, incid_lp I lAB && incid_lp I lCD) ->
exists J:Point, incid_lp J lAC && incid_lp J lBD.
```

for pg(3,3) : at least 40*40*40*40 = 2 560 000 cases to go

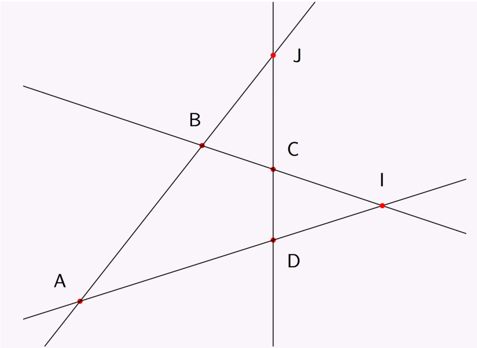
- Adding an order relation on points and lines.

```
Axiom a2 : forall A B C D:Point, forall lAB lCD lAC lBD :Line,

leP A B -> leP C D ->

dist_4p A B C D ->
incid_lp A lAB && incid_lp B lAB -> incid_lp C lCD && incid_lp D lCD ->
incid_lp A lAC && incid_lp C lAC -> incid_lp B lBD && incid_lp D lBD ->
(exists I:Point, incid_lp I lAB && incid_lp I lCD) ->
exists J:Point, incid_lp J lAC && incid_lp J lBD.
```

Without loss of generality

- Implementing a without loss of generality principle
  - Re-ordering points in a specific order
  - Re-using the previous statement (in a tactic)
  - it requires adapting the statement as follows :

```
... (exists I:Point, incid_lp I lAB && incid_lp I lCD) ->
(exists J:Point, (incid_lp J lAC && incid_lp J lBD)) /\
(exists K:Point, (incid_lp K lAD && incid_lp K lBC)).
```



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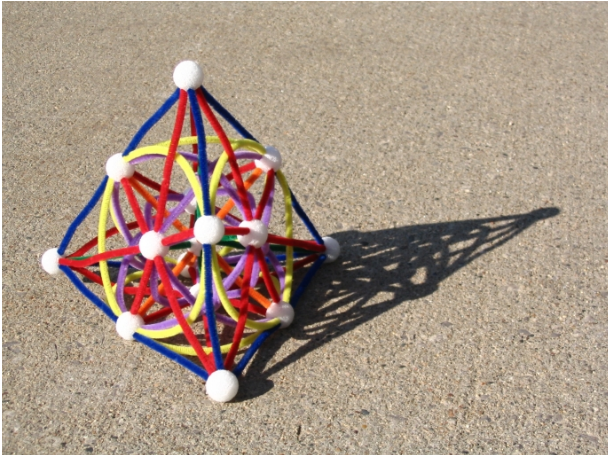
Results and Future Work

- Results
  - Using Intel(R) Core(TM) i5-4460 CPU @ 3.20GHz, 32 GB
  - 24 min to verify the axioms of projective geom. for pg(3,2)
  - 2 h to verify the axioms of projective geom. for pg(3,3)
  - some experiments with Z3 and lean
- Related and future work
  - Ranks (of sets of points) are an interesting alternative approach (PhD work of David Braun)
  - Next step : spreads and packings in pg(3,2)
  - Example of state-of-the-art results :
    Svetlana Topalova and Stela Zhelezova. *On transitive parallelisms of PG(3,4)*. 2017

Questions ?

- Thank you for your attention !



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