Exploring an Algorithm for Polynomial Interpolation in the *Theorema* System

Wolfgang Windsteiger

RISC Institute, University of Linz
A-4232 Hagenberg, Austria

1 Challenge for (Future) Mathematical Software Systems

AUTOMATE the entire process of

- inventing,
- proving, and
- applying

mathematical knowledge inside **ONE** system.

Theorema's key facilities for computer-support in these areas:

- Proving — computing — solving in **one logic** and **one software-technological frame**.
- **Domains and functors** for structured build-up of mathematical knowledge.
- Special proving, simplifying, and solving algorithms for **special mathematical theories**.
- **Software technological advances**:
    - Flexible syntax in textbook-style.
    - Proof presentation with high readability and postprocessing capabilities.
- **Knowledge management tools**:
    - Conjecture generation based on failure analysis.
    - Use of problem types, knowledge types and algorithm types in the theory exploration.
      (See Buchberger 2003, IMS'03, London.)

2 Case Study: Neville-Algorithm for Polynomial Interpolation

Case study consists of:

- Definition of the domain of **univariate polynomials by a functor**.
- Problem specification for **polynomial interpolation**.
- Formulation of a special solution of the problem, namely the **Neville algorithm**.
- **Correctness proof** for Neville algorithm.
- **Application of the algorithm** to concrete problems.

Case study should show:

- Problem specification, algorithms, proofs, and computation **in one system**.
- Formulation of mathematical knowledge in a generic way $\Longrightarrow$ **re-usability**.
- **Attractive syntax**:
    - Close to the common usage of notation in mathematical textbooks.
    - Formal rigor.
- **Didactic challenge**:
    - Every detail must be spelled out unambiguously.
    - Stay close to common use (and often ab-use) of mathematical notation.

Case study originates from the use of *Theorema* in the frame of two university lectures:

- **Algorithmic Methods**: emphasis on mathematical algorithms.
- **Predicate Logic as a Working Language**: emphasis on proving.

3 The Polynomial Functor

```
Needs["Theorema`"]
```

```
Definition["Polynomial Domain", any[K],
  Poly[K] := Functor[P, any[p, q, n, a],
    𝕤 = ⟨⟩
    ----------
    ∈_P[p] ⇔ ((p = ⟨0⟩_K)) ⋁ (is-tuple[p] ⋀ |p| > 0 ⋀_{i=1,...,|p|} ∈_K [p_i] ⋀ p_{|p|} ≠ 0_K))
    x_P := ⟨0_K, 1_K⟩
    nz-index_P[p] := { 1  ⇐  ⋀_{j=1,...,|p|} (p_j = 0_K)
                     { ∋_{i=1,...,|p|} ((p_i ≠ 0_K) ⋀_{j=i+1,...,|p|} (p_j = 0_K))  ⇐  otherwise
    canonic_P[p] := ⟨p_i | i=1,...,nz-index_P[p]⟩
    deg_P[p] := |p| − 1
    coef_P[p, n] := { p_{n+1}  ⇐  n ≥ 0 ⋀ n ≤ deg_P[p]
                    { 0_K      ⇐  otherwise
    const_P[a] := ⟨a⟩
    p +_P q := canonic_P[⟨coef_P[p, i] +_K coef_P[q, i] | i=0,...,Maximum[deg_P[p], deg_P[q]]⟩]
    p *_P q := ⟨Σ_{K, j=0,...,i} coef_P[p, j] *_K coef_P[q, i − j] | i=0,...,deg_P[p]+deg_P[q]⟩
    p /_P a := ⟨coef_P[p, i] /_K a | i=0,...,deg_P[p]⟩
    eval_P[p, a] := Σ_{K, i=0,...,deg_P[p]} coef_P[p, i] *_K a^i
    ]]
```

Taken from lecture notes.

```
Use[⟨Built-in["Tuples"], Built-in["Quantifiers"],
    Built-in["Numbers"], Built-in["Number Domains"],
    Built-in["Connectives"], Definition["Polynomial Domain"]⟩]
```

Compute$\left[\left\langle \frac{4}{3}, \frac{2}{5}, 0, \frac{1}{3}\right\rangle _{\overline{\text{Poly}}[\mathbb{Q}]} \left\langle 0, \frac{1}{2}\right\rangle\right]$

$$\left\langle \frac{4}{3}, -\frac{1}{10}, 0, \frac{1}{3}\right\rangle$$

Compute$\left[\left\langle \frac{4}{3}, \frac{2}{5}, 0, \frac{1}{3}\right\rangle _{\text{Poly}[\mathbb{Q}]}^{*} \left\langle 0, \frac{1}{30}, \frac{2}{13}, \frac{8}{7}\right\rangle\right]$

$$\left\langle 0, \frac{2}{45}, \frac{71}{325}, \frac{2164}{1365}, \frac{59}{126}, \frac{2}{39}, \frac{8}{21}\right\rangle$$

Compute$\left[\left\langle 0, \frac{1}{30}, \frac{2}{13}, \frac{8}{7}\right\rangle \Big/ _{\text{Poly}[\mathbb{Q}]} \frac{7}{3}\right]$

$$\left\langle 0, \frac{1}{70}, \frac{6}{91}, \frac{24}{49}\right\rangle$$

4 Problem Specification: Polynomial Interpolation

Given: x, a, K.
Find: p
such that: IsInterpolatingPolynomial[p, x, a, K], where

Definition["Interpolating polynomial: characterization", any[p, x, a, K],
IsInterpolatingPolynomial[p, x, a,
K] :⇔ $\left(\deg_{\text{Poly}[K]}[p] \leq |x| - 1 \bigwedge \forall_{i=1,\ldots,|x|} \left(\text{eval}_{\text{Poly}[K]}[p, x_i] = a_i\right)\right)$]

5 Neville's Algorithm

Algorithm["Neville", any[x, a, x0, $\overline{x}$, xn, a0, $\overline{a}$, an, K],

NevillePolynomial[⟨x⟩, ⟨a⟩, K] = $\underset{\text{Poly[K]}}{\text{const}}$[a]

NevillePolynomial[⟨x0, $\overline{x}$, xn⟩, ⟨a0, $\overline{a}$, an⟩, K] =

$$\left(\left(x_{\text{Poly[K]}}\ \overline{\text{Poly[K]}}\ \underset{\text{Poly[K]}}{\text{const}}[\text{x0}]\right) \underset{\text{Poly[K]}}{*} \text{NevillePolynomial}[\langle \overline{x}, \text{xn}\rangle, \langle \overline{a}, \text{an}\rangle, \text{K}]\ \overline{\text{Poly[K]}}\right.$$
$$\left.\left(x_{\text{Poly[K]}}\ \overline{\text{Poly[K]}}\ \underset{\text{Poly[K]}}{\text{const}}[\text{xn}]\right) \underset{\text{Poly[K]}}{*} \text{NevillePolynomial}[\langle \text{x0}, \overline{x}\rangle, \langle \text{a0}, \overline{a}\rangle, \text{K}]\right) \Big/_{\text{Poly[K]}}$$
$$\left(\text{xn}\ \overline{\text{K}}\ \text{x0}\right)$$

]

6 NevillePolynomial is Interpolating Polynomial

Theorem["Neville polynomial is interpolating polynomial",
any[is-tuple[x, a], is-field[K]],
with[|x| > 0 ∧ (|a| = |x|)],

IsInterpolatingPolynomial[**NevillePolynomial[x, a, K]**, x, a, K]]

7 Correctness Proof

7.1 Computer-generated Proof

See separate notebook.

7.2 Textbook Proof

See lecture notes.

8 Application of the Algorithm

```
UseAlso[Algorithm["Neville"]]
```

```
Compute[NevillePolynomial[⟨1, 2, 3, 4, 5⟩, ⟨3, 1, 5, 2, 6⟩, ℚ]]
```

$$\left\langle 51, -\frac{1093}{12}, \frac{443}{8}, -\frac{161}{12}, \frac{9}{8} \right\rangle$$

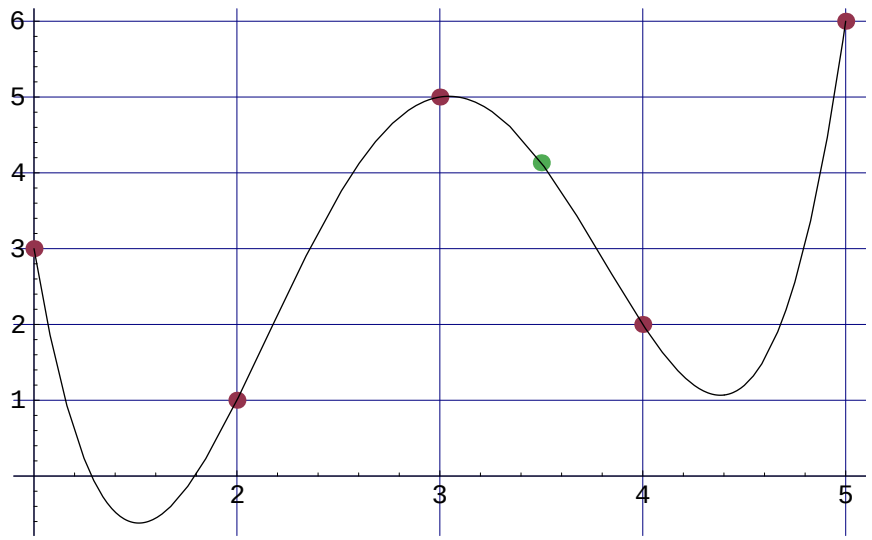
```
Compute[eval_Poly[ℚ][⟨51, -1093/12, 443/8, -161/12, 9/8⟩, 3]]
```

```
5
```

```
Compute[eval_Poly[ℚ][⟨51, -1093/12, 443/8, -161/12, 9/8⟩, 7/2]]
```

$$\frac{529}{128}$$

```
Plot[Compute[eval_Poly[ℚ][⟨51, -1093/12, 443/8, -161/12, 9/8⟩, x]],
    {x, 1, 5}, GridLines → Automatic, Prolog → {points, intermediate}];
```



9 Conclusion

- Continue case studies: apply *this prover* for proving e.g. correctness of **Horner's algorithm** for polynomial evaluation:

```
Algorithm["Polynomial Evaluation by Horner Scheme", any[p̄, c, a, K],
  Horner_Poly[K][⟨c⟩, a] := c
  Horner_Poly[K][⟨c, p̄⟩, a] := c +_K a *_K Horner_Poly[K][⟨p̄⟩, a]]
```

```
Theorem["Correctness Horner", any[is-tuple[p], a, K], with[|p| ≥ 1],
  Horner_Poly[K][p, a] = eval_Poly[K][p, a]]
```

- Continue case studies: apply *this prover* for proving **other recursive algorithms on tuples**.

- Problem specification, algorithms, proofs, and computation **can be done in one system**.

10 Textbook Proof

Proof: We choose K arbitrary but fixed and prove

$$\forall_{x,a} \deg_{\text{Poly}[K]}[\text{NevillePolynomial}[x, a, K]] \le |x| - 1 \bigwedge \forall_{i=1,\ldots,|x|} \left(\text{eval}_{\text{Poly}[K]}[\text{NevillePolynomial}[x, a, K], x_i] = a_i \right) \tag{1}$$

For proving (1) we use *induction over the length of the tuples* x and a.
Induction base: We show (1) for tuples of length 1, so we choose $x = \langle x_1 \rangle$ und $a = \langle a_1 \rangle$. We have to show

$$\deg_{\text{Poly}[K]}[\underbrace{\text{NevillePolynomial}[\langle x_1 \rangle, \langle a_1 \rangle, K]}_{\text{const}_{\text{Poly}[K]}[a_1]}] \le 0 \bigwedge \forall_{i=1,\ldots,1} \text{eval}_K\Big[\underbrace{\text{NevillePolynomial}[\langle x_1 \rangle, \langle a_1 \rangle, K]}_{\text{const}_{\text{Poly}[K]}[a_1]}, x_i\Big] = a_i$$

which, by the algorithm, simplifies to

$$\underbrace{\deg_{\text{Poly}[K]}\Big[\text{const}_{\text{Poly}[K]}[a_1]\Big] \le 0}_{\text{true}} \wedge \underbrace{\text{eval}_{\text{Poly}[K]}\Big[\text{const}_{\text{Poly}[K]}[a_1], x_1\Big] = a_1}_{\text{true}}$$

Induction hypothesis: For tuples x and a of length $n \ge 1$ let

$$\deg_{\text{Poly}[K]}[\text{NevillePolynomial}[x, a, K]] \le n - 1 \bigwedge \forall_{i=1,\ldots,n} \left(\text{eval}_{\text{Poly}[K]}[\text{NevillePolynomial}[x, a, K]][x_i] = a_i \right) \tag{2}$$

Induction conclusion: We show (1) for tuples of length $n+1$, so we choose $x = \langle \text{x0}, \overline{x}, \text{xn} \rangle$ and $a = \langle \text{a0}, \overline{a}, \text{an} \rangle$, where $\overline{x}$ and $\overline{a}$ are sequences of length greater equal 0. First we show:

$$\deg_{\text{Poly}[K]}\left[\text{NevillePolynomial}[\langle \text{x0}, \overline{x}, \text{xn}\rangle, \langle \text{a0}, \overline{a}, \text{an}\rangle, K]\right] =$$

$$\deg_{\text{Poly}[K]}\left[\frac{1}{\text{xn}-\text{x0}}\left((x-\text{x0})\underbrace{\text{NevillePolynomial}[\langle \overline{x}, \text{xn}\rangle, \langle \overline{a}, \text{an}\rangle, K]}_{\deg \le n-1} - (x-\text{xn})\underbrace{\text{NevillePolynomial}[\langle \text{x0}, \overline{x}\rangle, \langle \text{a0}, \overline{a}\rangle, K]}_{\deg \le n-1}\right)\right] \le n$$

It remains to show:

$$\forall_{i=1,\ldots,n+1}\left(\text{eval}_{\text{Poly}[K]}\left[\text{NevillePolynomial}[\langle \text{x0}, \overline{x}, \text{xn}\rangle, \langle \text{a0}, \overline{a}, \text{an}\rangle, K], x_i\right] = a_i\right)$$

By the algorithm, we have to show:

$$\forall_{i=1,\ldots,n+1}\ \text{eval}_{\text{Poly}[K]}\left[\frac{1}{\text{xn}-\text{x0}}\left((x-\text{x0})\,\text{NevillePolynomial}[\langle \overline{x}, \text{xn}\rangle, \langle \overline{a}, \text{an}\rangle, K] - (x-\text{xn})\,\text{NevillePolynomial}[\langle \text{x0}, \overline{x}\rangle, \langle \text{a0}, \overline{a}\rangle, K]\right), x_i\right] = a_i$$

By basic properties of polynomial evaluation, we need to prove:

$$\forall_{i=1,\ldots,n+1}\ \frac{1}{\text{xn}-\text{x0}}\left(\text{eval}_{\text{Poly}[K]}\left[x-\text{x0}, x_i\right]\text{eval}_{\text{Poly}[K]}\left[\text{NevillePolynomial}[\langle \overline{x}, \text{xn}\rangle, \langle \overline{a}, \text{an}\rangle, K], x_i\right] - \text{eval}_{\text{Poly}[K]}\left[x-\text{xn}, x_i\right]\text{eval}_{\text{Poly}[K]}\left[\text{NevillePolynomial}[\langle \text{x0}, \overline{x}\rangle, \langle \text{a0}, \overline{a}\rangle, K], x_i\right]\right) = a_i$$

We distinguish 3 cases:

- $i = 1$:

$$\frac{1}{\text{xn}-\text{x0}}\left(\overbrace{\text{eval}_{\text{Poly}[K]}\left[x-\text{x0}, \text{x0}\right]}^{=0}\text{eval}_{\text{Poly}[K]}\left[\text{NevillePolynomial}[\langle \overline{x}, \text{xn}\rangle, \langle \overline{a}, \text{an}\rangle, K], \text{x0}\right] - \underbrace{\text{eval}_{\text{Poly}[K]}\left[x-\text{xn}, \text{x0}\right]}_{=\text{x0}-\text{xn}}\underbrace{\text{eval}_{\text{Poly}[K]}\left[\text{NevillePolynomial}[\langle \text{x0}, \overline{x}\rangle, \langle \text{a0}, \overline{a}\rangle, K], \text{x0}\right]}_{=\text{a0 (by induction hypothesis)}}\right) = \text{a0}$$

- $i = n + 1$:

$$\frac{1}{\text{xn} - \text{x0}} \left(\overbrace{\underset{\text{Poly}[K]}{\text{eval}}\,[x - \text{x0}, \text{xn}]}^{=\text{xn}-\text{x0}} \overbrace{\underset{\text{Poly}[K]}{\text{eval}}\,[\text{NevillePolynomial}[\langle \overline{x}, \text{xn} \rangle, \langle \overline{a}, \text{an} \rangle, K], \text{xn}]}^{=\text{an (by induction hypothesis)}} - \underbrace{\underset{\text{Poly}[K]}{\text{eval}}\,[x - \text{xn}, \text{xn}]}_{=0} \underset{\text{Poly}[K]}{\text{eval}}\,[\text{NevillePolynomial}[\langle \text{x0}, \overline{x} \rangle, \langle \text{a0}, \overline{a} \rangle, K], \text{xn}] \right) = \text{an}$$

- $1 < i < n + 1$:

$$\frac{1}{\text{xn} - \text{x0}} \left(\overbrace{\underset{\text{Poly}[K]}{\text{eval}}\,[x - \text{x0},\, x_i]}^{=x_i - \text{x0}} \overbrace{\underset{\text{Poly}[K]}{\text{eval}}\,[\text{NevillePolynomial}[\langle \overline{x},\, \text{xn}\rangle,\, \langle \overline{a},\, \text{an}\rangle,\, K],\, x_i]}^{=a_i \text{ (by induction hypothesis)}} - \right.$$

$$\left. \underbrace{\underset{\text{Poly}[K]}{\text{eval}}\,[x - \text{xn},\, x_i]}_{=x_i - \text{xn}} \underbrace{\underset{\text{Poly}[K]}{\text{eval}}\,[\text{NevillePolynomial}[\langle \text{x0},\, \overline{x}\rangle,\, \langle \text{a0},\, \overline{a}\rangle,\, K],\, x_i]}_{a_i \text{ (by induction hypothesis)}} \right) =$$

$$\frac{1}{\text{xn} - \text{x0}} \left((x_i - \text{x0} - x_i + \text{xn})\, a_i \right) = a_i$$

□

11 Computer-generated Proof

Proposition["Polynomial Domain: knowledge", any[K, a, b, x],

$\underset{\text{Poly}[K]}{\text{eval}}\left[\underset{\text{Poly}[K]}{\text{const}}[\text{a}],\, \text{x}\right] = \text{a}$	"eval const"
$\underset{\text{Poly}[K]}{\text{deg}}\left[\underset{\text{Poly}[K]}{\text{const}}[\text{a}]\right] = 0$	"deg const"
$\underset{\text{Poly}[K]}{\text{eval}}\left[\text{a} \underset{\text{Poly}[K]}{/} \text{b},\, \text{x}\right] = \underset{\text{Poly}[K]}{\text{eval}}\,[\text{a},\, \text{x}] \Big/ \underset{K}{\text{b}}$	"eval /"
$\underset{\text{Poly}[K]}{\text{eval}}\left[\text{a} \underset{\text{Poly}[K]}{\bar{-}} \text{b},\, \text{x}\right] = \underset{\text{Poly}[K]}{\text{eval}}\,[\text{a},\, \text{x}] \,\bar{-}_K\, \underset{\text{Poly}[K]}{\text{eval}}\,[\text{b},\, \text{x}]$	"eval −"
$\underset{\text{Poly}[K]}{\text{eval}}\left[\text{a} \underset{\text{Poly}[K]}{*} \text{b},\, \text{x}\right] = \underset{\text{Poly}[K]}{\text{eval}}\,[\text{a},\, \text{x}] *_K \underset{\text{Poly}[K]}{\text{eval}}\,[\text{b},\, \text{x}]$	"eval ∗"]
$\underset{\text{Poly}[K]}{\text{eval}}\left[\underset{\text{Poly}[K]}{x},\, \text{x}\right] = \text{x}$	"eval x"
$\underset{\text{Poly}[K]}{\text{deg}}\left[\text{a} \underset{\text{Poly}[K]}{/} \text{b}\right] = \underset{\text{Poly}[K]}{\text{deg}}\,[\text{a}]$	"deg /"
$\underset{\text{Poly}[K]}{\text{deg}}\,[\text{a}] \le \text{x} \bigwedge \underset{\text{Poly}[K]}{\text{deg}}\,[\text{b}] \le \text{x} \Rightarrow \underset{\text{Poly}[K]}{\text{deg}}\left[\text{a} \underset{\text{Poly}[K]}{\bar{-}} \text{b}\right] \le \text{x}$	"deg −"
$\underset{\text{Poly}[K]}{\text{deg}}\,[\text{a}] \le \text{x} - 1 \Rightarrow \underset{\text{Poly}[K]}{\text{deg}}\left[\left(\underset{\text{Poly}[K]}{x} \underset{\text{Poly}[K]}{\bar{-}} \underset{\text{Poly}[K]}{\text{const}}[\text{b}]\right) \underset{\text{Poly}[K]}{*} \text{a}\right] \le \text{x}$	"deg x∗(.)"

Proposition["Fields: knowledge", any[K, a, b, c],

$a \underset{K}{+} a = \underset{K}{0}$	"Inv +"
$\underset{K}{0} \underset{K}{*} a = \underset{K}{0}$	"0 ∗"
$\underset{K}{0} \underset{K}{+} a = \underset{K}{-} a$	"Neut + "
$a \underset{K}{-} \underset{K}{0} = a$	"Neut − "
$\underset{K}{-}\left(\left(a \underset{K}{-} b\right) \underset{K}{*} c\right) \Big/ \underset{K}{\left(b \underset{K}{-} a\right)} = c$	"Inv ∗ neg"]
$\left(a \underset{K}{*} c\right) \Big/ \underset{K}{a} = c$	"Inv ∗"
$a \underset{K}{*} c \underset{K}{-} b \underset{K}{*} c = \left(a \underset{K}{-} b\right) \underset{K}{*} c$	"Dist −"
$\left(a \underset{K}{-} b\right) \underset{K}{-} \left(a \underset{K}{-} c\right) = c \underset{K}{-} b$	"−(.)"

Theory["Interpolation",

Algorithm["Neville"]
Definition["Interpolating polynomial: characterization"]
Proposition["Polynomial Domain: knowledge"]
Proposition["Fields: knowledge"]]

Prove[Theorem["Neville polynomial is interpolating polynomial"],
using → Theory["Interpolation"],
by → TupleProver,
built-in → ⟨Built-in["Tuples"], Built-in["Quantifiers"], Built-in["Numbers"]⟩,
ProverOptions → {SimplifyFormula → False, RWInnermost → False, ApplyBuiltIns → {"goal"},
GRWTarget → {"goal", "kb"}, KBReorderGoalPattern → NevillePolynomial[_, _, _],
ModusPonensException → •[_[deg][_] ≤ _]}, SearchDepth → 100]

Prove:

(Theorem (Neville polynomial is interpolating polynomial))

$$\underset{\substack{\text{IsTuple}[x]\\ \text{IsTuple}[a]\\ \text{is-field}[K]\\ |x|>0\wedge(|a|=|x|)}}{\forall} \text{IsInterpolatingPolynomial}[\text{NevillePolynomial}[x, a, K], x, a, K],$$

For proving (Theorem (Neville polynomial is interpolating polynomial)), we choose K_0 with

(1) is-field[K_0]

arbitrary but fixed and show

(2) $$\underset{\substack{\text{IsTuple}[x]\\ \text{IsTuple}[a]\\ |x|>0\wedge(|a|=|x|)}}{\forall} \text{IsInterpolatingPolynomial}[\text{NevillePolynomial}[x, a, K_0], x, a, K_0].$$

We prove (2) by induction over the length of x:

Induction base: We prove the goal for all tuples of length 1, hence we prove

(5) $\forall_{a}\ (\text{IsTuple}[a] \wedge (|a| = 1) \Rightarrow$:

$$\text{IsInterpolatingPolynomial}[\text{NevillePolynomial}[\langle x_0\rangle, a, K_0], \langle x_0\rangle, a, K_0])$$

For proving (5), we choose an arbitrary tuple of length 1 , i.e. we have to show

(13) $\text{IsInterpolatingPolynomial}[\text{NevillePolynomial}[\langle x_0\rangle, \langle a_0\rangle, K_0], \langle x_0\rangle, \langle a_0\rangle, K_0]$.

Formula (13), using (Definition (Interpolating polynomial: characterization)), is implied by:

$$\forall_{i=1,\ldots,|\langle x_0\rangle|} \left(\underset{\text{Poly}[K_0]}{\text{eval}}\ [\text{NevillePolynomial}[\langle x_0\rangle, \langle a_0\rangle, K_0], (\langle x_0\rangle)_i] = (\langle a_0\rangle)_i \right) \bigwedge ,$$

$$\underset{\text{Poly}[K_0]}{\deg}\ [\text{NevillePolynomial}[\langle x_0\rangle, \langle a_0\rangle, K_0]] \leq -1 + |\langle x_0\rangle|$$

which, using (Algorithm (Neville): 1), is implied by:

$$\forall_{i=1,\ldots,|\langle x_0\rangle|} \left(\underset{\text{Poly}[K_0]}{\text{eval}} \left[\underset{\text{Poly}[K_0]}{\text{const}}\ [a_0], (\langle x_0\rangle)_i \right] = (\langle a_0\rangle)_i \right) \bigwedge \underset{\text{Poly}[K_0]}{\deg} \left[\underset{\text{Poly}[K_0]}{\text{const}}\ [a_0] \right] \leq -1 + |\langle x_0\rangle| ,$$

which, using (Definition (Polynomial Domain: knowledge): eval const), is implied by:

$$\forall_{i=1,\ldots,|\langle x_0\rangle|} (a_0 = (\langle a_0\rangle)_i) \bigwedge \underset{\text{Poly}[K_0]}{\deg} \left[\underset{\text{Poly}[K_0]}{\text{const}}\ [a_0] \right] \leq -1 + |\langle x_0\rangle| ,$$

which, using (Definition (Polynomial Domain: knowledge): deg const), is implied by:

(14) $$\forall_{i=1,\ldots,|\langle x_0\rangle|} (a_0 = (\langle a_0\rangle)_i) \bigwedge 0 \leq -1 + |\langle x_0\rangle| .$$

Using available computation rules we evaluate (14):

(15) True .

Formula (15) is true because it is the constant True.

Induction hypothesis: For an arbitrary

(6) $n_0 \geq 1$

we assume

(7) $\forall_{x} \left((|x| = n_0) \Rightarrow \forall_{a}\ (\text{IsTuple}[a] \wedge (|a| = n_0) \Rightarrow \right.$.

$$\left. \text{IsInterpolatingPolynomial}[\text{NevillePolynomial}[x, a, K_0], x, a, K_0]) \right)$$

Induction step: Induction step: We choose an arbitrary but fixed x_1 with

(9) $\text{IsTuple}[x_1]$ and

(8) $|x_1| = 1 + n_0$

and show

(10) $\forall_{a}\ (\text{IsTuple}[a] \wedge (|a| = 1 + n_0) \Rightarrow$:

$$\text{IsInterpolatingPolynomial}[\text{NevillePolynomial}[x_1, a, K_0], x_1, a, K_0])$$

For proving (10), we choose an arbitrary tuple of length $1 + n_0$, i.e. we assume

(17) $\text{IsTuple}[a_1]$ and

(16) $|a_1| = 1 + n_0$

and show

(18) $\text{IsInterpolatingPolynomial}[\text{NevillePolynomial}[x_1, a_1, K_0], x_1, a_1, K_0]$.

From formula (Algorithm (Neville): 2) and knowledge about the length of tuple constants in the goal we guess a representation of the tuple constants in terms of concrete tuples:

$$a_1 \to \langle a0_0, \overline{a_0}, an_0\rangle$$

$$x_1 \to \langle x0_0, \overline{x_0}, xn_0\rangle$$

Thus, we need to show

(21)

IsInterpolatingPolynomial[
NevillePolynomial[⟨$x0_0$, $\overline{x_0}$, xn_0⟩, ⟨$a0_0$, $\overline{a_0}$, an_0⟩, K_0], ⟨$x0_0$, $\overline{x_0}$, xn_0⟩, ⟨$a0_0$, $\overline{a_0}$, an_0⟩, K_0] .

From what we already know follows:

From (25) we can infer

(28) $|\langle \overline{a_0}, an_0 \rangle| = n_0$,

(29) $|\langle a0_0, \overline{a_0} \rangle| = n_0$.

From (27) we can infer

(30) $|\langle \overline{x_0}, xn_0 \rangle| = n_0$,

(31) $|\langle x0_0, \overline{x_0} \rangle| = n_0$.

From what we already know follows:

From (28) we can infer

(34) $|\langle \overline{a_0} \rangle| = -1 + n_0$.

From (30) we can infer

(36) $|\langle \overline{x_0} \rangle| = -1 + n_0$.

From (28) together with (6) we know

(39) $|\langle \overline{a_0}, an_0 \rangle| \geq 1$.

From (29) together with (6) we know

(41) $|\langle a0_0, \overline{a_0} \rangle| \geq 1$.

From (7) together with (30) we know

(42)

$\forall_a$ (IsTuple[a] $\wedge$ ($|a| = n_0$) $\Rightarrow$
IsInterpolatingPolynomial[NevillePolynomial[⟨$\overline{x_0}$, xn_0⟩, a, K_0], ⟨$\overline{x_0}$, xn_0⟩, a, K_0]) .

From (30) together with (6) we know

(43) $|\langle \overline{x_0}, xn_0 \rangle| \geq 1$.

From (7) together with (31) we know

(44)

$\forall_a$ (IsTuple[a] $\wedge$ ($|a| = n_0$) $\Rightarrow$
IsInterpolatingPolynomial[NevillePolynomial[⟨$x0_0$, $\overline{x_0}$⟩, a, K_0], ⟨$x0_0$, $\overline{x_0}$⟩, a, K_0]) .

From (31) together with (6) we know

(45) $|\langle x0_0, \overline{x_0} \rangle| \geq 1$.

From what we already know follows:

From (42) together with (28) we know

(58) IsInterpolatingPolynomial[
NevillePolynomial[⟨$\overline{x_0}$, xn_0⟩, ⟨$\overline{a_0}$, an_0⟩, K_0], ⟨$\overline{x_0}$, xn_0⟩, ⟨$\overline{a_0}$, an_0⟩, K_0] .

From (44) together with (29) we know

(65) IsInterpolatingPolynomial[
NevillePolynomial[⟨$x0_0$, $\overline{x_0}$⟩, ⟨$a0_0$, $\overline{a_0}$⟩, K_0], ⟨$x0_0$, $\overline{x_0}$⟩, ⟨$a0_0$, $\overline{a_0}$⟩, K_0] .

Formula (21), using (Definition (Interpolating polynomial: characterization)), is implied by:

$$\forall_{i=1,\ldots,|\langle x0_0,\overline{x_0},xn_0\rangle|}\left(\operatorname{eval}_{\mathrm{Poly}[K_0]}[\mathrm{NevillePolynomial}[\langle x0_0,\overline{x_0},xn_0\rangle,\langle a0_0,\overline{a_0},an_0\rangle,K_0],\quad,\right.$$

$$\left.(\langle x0_0,\overline{x_0},xn_0\rangle)_i]=(\langle a0_0,\overline{a_0},an_0\rangle)_i\right)\bigwedge$$

$$\deg_{\mathrm{Poly}[K_0]}[\mathrm{NevillePolynomial}[\langle x0_0,\overline{x_0},xn_0\rangle,\langle a0_0,\overline{a_0},an_0\rangle,K_0]]\le -1+|\langle x0_0,\overline{x_0},xn_0\rangle|$$

which, using (27), is implied by:

$$\forall_{i=1,\ldots,1+n_0}\left(\operatorname{eval}_{\mathrm{Poly}[K_0]}[\mathrm{NevillePolynomial}[\langle x0_0,\overline{x_0},xn_0\rangle,\langle a0_0,\overline{a_0},an_0\rangle,K_0],(\langle x0_0,\overline{x_0},xn_0\rangle)_i]=,\right.$$

$$\left.(\langle a0_0,\overline{a_0},an_0\rangle)_i\right)\bigwedge$$

$$\deg_{\mathrm{Poly}[K_0]}[\mathrm{NevillePolynomial}[\langle x0_0,\overline{x_0},xn_0\rangle,\langle a0_0,\overline{a_0},an_0\rangle,K_0]]\le -1+(1+n_0)$$

which, using (Algorithm (Neville): 2), is implied by:

$$\underset{i=1,\ldots,1+n_0}{\forall}\left(\underset{\text{Poly}[K_0]}{\text{eval}}\left[\left(\left(\left(x_{\text{Poly}[K_0]}\,\overline{\text{Poly}}[K_0]\,\underset{\text{Poly}[K_0]}{\text{const}}[x0_0]\right)\underset{\text{Poly}[K_0]}{*}\text{NevillePolynomial}[\langle\overline{x_0},\, xn_0\rangle,\, \langle\overline{a_0},\, an_0\rangle,\, K_0]\right)_{\overline{\text{Poly}}[K_0]}\left(x_{\text{Poly}[K_0]}\,\overline{\text{Poly}}[K_0]\,\underset{\text{Poly}[K_0]}{\text{const}}[xn_0]\right)\underset{\text{Poly}[K_0]}{*}\text{NevillePolynomial}[\langle x0_0,\, \overline{x_0}\rangle,\, \langle a0_0,\, \overline{a_0}\rangle,\, K_0]\right)\Big/_{\text{Poly}[K_0]}\left(xn_0\;{}_{\overline{K_0}}\;x0_0\right),\, (\langle x0_0,\, \overline{x_0},\, xn_0\rangle)_i\Big] = (\langle a0_0,\, \overline{a_0},\, an_0\rangle)_i\right)\bigwedge$$

$$\deg_{\text{Poly}[K_0]}\left[\left(\left(\left(x_{\text{Poly}[K_0]}\,\overline{\text{Poly}}[K_0]\,\underset{\text{Poly}[K_0]}{\text{const}}[x0_0]\right)\underset{\text{Poly}[K_0]}{*}\text{NevillePolynomial}[\langle\overline{x_0},\, xn_0\rangle,\, \langle\overline{a_0},\, an_0\rangle,\, K_0]\right)_{\overline{\text{Poly}}[K_0]}\left(x_{\text{Poly}[K_0]}\,\overline{\text{Poly}}[K_0]\,\underset{\text{Poly}[K_0]}{\text{const}}[xn_0]\right)\underset{\text{Poly}[K_0]}{*}\text{NevillePolynomial}[\langle x0_0,\, \overline{x_0}\rangle,\, \langle a0_0,\, \overline{a_0}\rangle,\, K_0]\right)\Big/_{\text{Poly}[K_0]}\left(xn_0\;{}_{\overline{K_0}}\;x0_0\right)\Big] \le -1+(1+n_0)$$

,

which, using (Definition (Polynomial Domain: knowledge): eval /), is implied by:

$$\underset{i=1,\ldots,1+n_0}{\forall}\left(\underset{\text{Poly}[K_0]}{\text{eval}}\left[\left(\underset{\text{Poly}[K_0]}{x}\,\underset{\text{Poly}[K_0]}{\overline{\text{Poly}}}\,\underset{\text{Poly}[K_0]}{\text{const}}\,[x0_0]\right)\underset{\text{Poly}[K_0]}{*}\right.\right.$$

$$\text{NevillePolynomial}[\langle\overline{x_0},\, xn_0\rangle,\, \langle\overline{a_0},\, an_0\rangle,\, K_0]\,\underset{\text{Poly}[K_0]}{\overline{\text{Poly}}}$$

$$\left(\underset{\text{Poly}[K_0]}{x}\,\underset{\text{Poly}[K_0]}{\overline{\text{Poly}}}\,\underset{\text{Poly}[K_0]}{\text{const}}\,[xn_0]\right)\underset{\text{Poly}[K_0]}{*}\text{NevillePolynomial}[\langle x0_0,\, \overline{x_0}\rangle,\, \langle a0_0,\, \overline{a_0}\rangle,\, K_0],$$

$$\left.\left.(\langle x0_0,\, \overline{x_0},\, xn_0\rangle)_i\right]\Big/\underset{K_0}{\left(xn_0\ \overline{K_0}\ x0_0\right)} = (\langle a0_0,\, \overline{a_0},\, an_0\rangle)_i\right)\bigwedge$$

$$\underset{\text{Poly}[K_0]}{\deg}\left[\left(\left(\underset{\text{Poly}[K_0]}{x}\,\underset{\text{Poly}[K_0]}{\overline{\text{Poly}}}\,\underset{\text{Poly}[K_0]}{\text{const}}\,[x0_0]\right)\underset{\text{Poly}[K_0]}{*}\text{NevillePolynomial}[\langle\overline{x_0},\, xn_0\rangle,\, \langle\overline{a_0},\, an_0\rangle,\, K_0]\,\underset{\text{Poly}[K_0]}{\overline{\text{Poly}}}\right.\right.$$

$$\left(\underset{\text{Poly}[K_0]}{x}\,\underset{\text{Poly}[K_0]}{\overline{\text{Poly}}}\,\underset{\text{Poly}[K_0]}{\text{const}}\,[xn_0]\right)\underset{\text{Poly}[K_0]}{*}$$

$$\left.\left.\text{NevillePolynomial}[\langle x0_0,\, \overline{x_0}\rangle,\, \langle a0_0,\, \overline{a_0}\rangle,\, K_0]\right)\Big/\underset{\text{Poly}[K_0]}{\left(xn_0\ \overline{K_0}\ x0_0\right)}\right] \le -1 + (1 + n_0)$$

,

which, using (Definition (Polynomial Domain: knowledge): eval −), is implied by:

$$\underset{i=1,\ldots,1+n_0}{\forall}\left(\left(\underset{\text{Poly}[K_0]}{\text{eval}}\left[\left(\underset{\text{Poly}[K_0]}{x}\ \underset{\text{Poly}[K_0]}{\overline{\text{Poly}}}\ \underset{\text{Poly}[K_0]}{\text{const}}\ [x0_0]\right)^{*}_{\text{Poly}[K_0]}\right.\right.\right.$$

$$\text{NevillePolynomial}[\langle \overline{x_0},\, xn_0\rangle,\, \langle \overline{a_0},\, an_0\rangle,\, K_0],\, (\langle x0_0,\, \overline{x_0},\, xn_0\rangle)_i\Big]\ \underset{\overline{K_0}}{}\ \underset{\text{Poly}[K_0]}{\text{eval}}\Big[$$

$$\left(\underset{\text{Poly}[K_0]}{x}\ \underset{\text{Poly}[K_0]}{\overline{\text{Poly}}}\ \underset{\text{Poly}[K_0]}{\text{const}}\ [xn_0]\right)^{*}_{\text{Poly}[K_0]}\text{NevillePolynomial}[\langle x0_0,\, \overline{x_0}\rangle,\, \langle a0_0,\, \overline{a_0}\rangle,\, K_0],$$

$$\left.\left.\left.(\langle x0_0,\, \overline{x_0},\, xn_0\rangle)_i\Big]\right)\Big/_{K_0}\left(xn_0\ \underset{\overline{K_0}}{}\ x0_0\right) = (\langle a0_0,\, \overline{a_0},\, an_0\rangle)_i\right)\Bigg/\bigwedge$$

$$\underset{\text{Poly}[K_0]}{\deg}\Big[\Big(\Big(\Big(\underset{\text{Poly}[K_0]}{x}\ \underset{\text{Poly}[K_0]}{\overline{\text{Poly}}}\ \underset{\text{Poly}[K_0]}{\text{const}}\ [x0_0]\Big)^{*}_{\text{Poly}[K_0]}\text{NevillePolynomial}[\langle \overline{x_0},\, xn_0\rangle,\, \langle \overline{a_0},\, an_0\rangle,\, K_0]\ \underset{\text{Poly}[K_0]}{\overline{\text{Poly}}}$$

$$\left(\underset{\text{Poly}[K_0]}{x}\ \underset{\text{Poly}[K_0]}{\overline{\text{Poly}}}\ \underset{\text{Poly}[K_0]}{\text{const}}\ [xn_0]\right)^{*}_{\text{Poly}[K_0]}$$

$$\left.\text{NevillePolynomial}[\langle x0_0,\, \overline{x_0}\rangle,\, \langle a0_0,\, \overline{a_0}\rangle,\, K_0]\Big)\Big/_{\text{Poly}[K_0]}\left(xn_0\ \underset{\overline{K_0}}{}\ x0_0\right)\right] \le -1 + (1 + n_0)$$

,

which, using ⟨Definition (Polynomial Domain: knowledge): eval *⟩, is implied by:

$$\forall_{i=1,\ldots,1+n_0}\left(\left(\left(\operatorname{eval}_{\mathrm{Poly}[K_0]}\left[\underset{\mathrm{Poly}[K_0]}{x}{}_{\mathrm{Poly}[K_0]}\overline{\mathrm{Poly}[K_0]}\operatorname{const}_{\mathrm{Poly}[K_0]}[x0_0],\ (\langle x0_0,\ \overline{x_0},\ xn_0\rangle)_i\right]\underset{K_0}{*}\right.\right.\right.$$

$$\operatorname{eval}_{\mathrm{Poly}[K_0]}[\mathrm{NevillePolynomial}[\langle\overline{x_0},\ xn_0\rangle,\ \langle\overline{a_0},\ an_0\rangle,\ K_0],\ (\langle x0_0,\ \overline{x_0},\ xn_0\rangle)_i]\ \underset{\overline{K_0}}{}$$

$$\operatorname{eval}_{\mathrm{Poly}[K_0]}\left[\underset{\mathrm{Poly}[K_0]}{x}{}_{\mathrm{Poly}[K_0]}\overline{\mathrm{Poly}[K_0]}\operatorname{const}_{\mathrm{Poly}[K_0]}[xn_0],\ (\langle x0_0,\ \overline{x_0},\ xn_0\rangle)_i\right]\underset{K_0}{*}$$

$$\left.\left.\left.\operatorname{eval}_{\mathrm{Poly}[K_0]}[\mathrm{NevillePolynomial}[\langle x0_0,\ \overline{x_0}\rangle,\ \langle a0_0,\ \overline{a_0}\rangle,\ K_0],\ (\langle x0_0,\ \overline{x_0},\ xn_0\rangle)_i]\right)\middle/\underset{K_0}{}\right.\right.$$

$$\left.\left.\left(xn_0\ \underset{\overline{K_0}}{}\ x0_0\right) = (\langle a0_0,\ \overline{a_0},\ an_0\rangle)_i\right)\right)\bigwedge$$

$$\deg_{\mathrm{Poly}[K_0]}\left[\left(\left(\left(\underset{\mathrm{Poly}[K_0]}{x}{}_{\mathrm{Poly}[K_0]}\overline{\mathrm{Poly}[K_0]}\operatorname{const}_{\mathrm{Poly}[K_0]}[x0_0]\right)\underset{\mathrm{Poly}[K_0]}{*}\mathrm{NevillePolynomial}[\langle\overline{x_0},\ xn_0\rangle,\ \langle\overline{a_0},\ an_0\rangle,\ K_0]\ \underset{\mathrm{Poly}[K_0]}{}\right.\right.\right.$$

$$\left.\left(\underset{\mathrm{Poly}[K_0]}{x}{}_{\mathrm{Poly}[K_0]}\overline{\mathrm{Poly}[K_0]}\operatorname{const}_{\mathrm{Poly}[K_0]}[xn_0]\right)\underset{\mathrm{Poly}[K_0]}{*}\right.$$

$$\left.\left.\left.\mathrm{NevillePolynomial}[\langle x0_0,\ \overline{x_0}\rangle,\ \langle a0_0,\ \overline{a_0}\rangle,\ K_0]\right)\middle/\underset{\mathrm{Poly}[K_0]}{}\left(xn_0\ \underset{\overline{K_0}}{}\ x0_0\right)\right]\leq -1+(1+n_0)$$

,

which, using ⟨Definition (Polynomial Domain: knowledge): eval −⟩, is implied by:

$$\underset{i=1,\ldots,1+n_0}{\forall}\left(\left(\left(\left(\underset{\text{Poly}[K_0]}{\text{eval}}\left[\underset{\text{Poly}[K_0]}{x},\ (\langle x0_0,\ \overline{x_0},\ xn_0\rangle)_i\right]\ \underset{K_0}{\overline{\ }}\ \underset{\text{Poly}[K_0]}{\text{eval}}\left[\underset{\text{Poly}[K_0]}{\text{const}}[x0_0],\ (\langle x0_0,\ \overline{x_0},\ xn_0\rangle)_i\right]\right)\underset{K_0}{*}\right.\right.\right.$$
$$\underset{\text{Poly}[K_0]}{\text{eval}}[\text{NevillePolynomial}[\langle\overline{x_0},\ xn_0\rangle,\ \langle\overline{a_0},\ an_0\rangle,\ K_0],\ (\langle x0_0,\ \overline{x_0},\ xn_0\rangle)_i]\ \underset{K_0}{\overline{\ }}$$
$$\left(\underset{\text{Poly}[K_0]}{\text{eval}}\left[\underset{\text{Poly}[K_0]}{x},\ (\langle x0_0,\ \overline{x_0},\ xn_0\rangle)_i\right]\ \underset{K_0}{\overline{\ }}\ \underset{\text{Poly}[K_0]}{\text{eval}}\left[\underset{\text{Poly}[K_0]}{\text{const}}[xn_0],\ (\langle x0_0,\ \overline{x_0},\ xn_0\rangle)_i\right]\right)\underset{K_0}{*}$$
$$\left.\left.\left.\underset{\text{Poly}[K_0]}{\text{eval}}[\text{NevillePolynomial}[\langle x0_0,\ \overline{x_0}\rangle,\ \langle a0_0,\ \overline{a_0}\rangle,\ K_0],\ (\langle x0_0,\ \overline{x_0},\ xn_0\rangle)_i]\right)\Big/_{K_0}\right.\right.$$
$$\left.\left.\left(xn_0\ \underset{K_0}{\overline{\ }}\ x0_0\right) = (\langle a0_0,\ \overline{a_0},\ an_0\rangle)_i\right)\right)\bigwedge$$
$$\underset{\text{Poly}[K_0]}{\deg}\left[\left(\left(\left(\underset{\text{Poly}[K_0]}{x}\ \underset{\text{Poly}[K_0]}{\overline{\ }}\ \underset{\text{Poly}[K_0]}{\text{const}}[x0_0]\right)\underset{\text{Poly}[K_0]}{*}\text{NevillePolynomial}[\langle\overline{x_0},\ xn_0\rangle,\ \langle\overline{a_0},\ an_0\rangle,\ K_0]\ \underset{\text{Poly}[K_0]}{\overline{\text{Poly}}[K_0]}\right.\right.\right.$$
$$\left(\underset{\text{Poly}[K_0]}{x}\ \underset{\text{Poly}[K_0]}{\overline{\ }}\ \underset{\text{Poly}[K_0]}{\text{const}}[xn_0]\right)\underset{\text{Poly}[K_0]}{*}$$
$$\left.\left.\left.\text{NevillePolynomial}[\langle x0_0,\ \overline{x_0}\rangle,\ \langle a0_0,\ \overline{a_0}\rangle,\ K_0]\right)\Big/_{\text{Poly}[K_0]}\left(xn_0\ \underset{K_0}{\overline{\ }}\ x0_0\right)\right]\le -1+(1+n_0)\right)$$

,

which, using (Definition (Polynomial Domain: knowledge): eval const), is implied by:

$$\forall_{i=1,\ldots,1+n_0}\left(\left(\left(\left(\operatorname{eval}_{\text{Poly}[K_0]}\left[\underset{\text{Poly}[K_0]}{x},\ (\langle x0_0,\ \overline{x_0},\ xn_0\rangle)_i\right]_{\overline{K_0}}\ x0_0\right)^*_{K_0}\right.\right.\right.$$
$$\operatorname{eval}_{\text{Poly}[K_0]}[\text{NevillePolynomial}[\langle \overline{x_0},\ xn_0\rangle,\ \langle \overline{a_0},\ an_0\rangle,\ K_0],\ (\langle x0_0,\ \overline{x_0},\ xn_0\rangle)_i]_{\overline{K_0}}$$
$$\left(\operatorname{eval}_{\text{Poly}[K_0]}\left[\underset{\text{Poly}[K_0]}{x},\ (\langle x0_0,\ \overline{x_0},\ xn_0\rangle)_i\right]_{\overline{K_0}}\ xn_0\right)^*_{K_0}$$
$$\left.\left.\left.\operatorname{eval}_{\text{Poly}[K_0]}[\text{NevillePolynomial}[\langle x0_0,\ \overline{x_0}\rangle,\ \langle a0_0,\ \overline{a_0}\rangle,\ K_0],\ (\langle x0_0,\ \overline{x_0},\ xn_0\rangle)_i]\right)\Big/_{K_0}\right.\right.$$
$$\left.\left.\left(xn_0\ {}_{\overline{K_0}}\ x0_0\right) = (\langle a0_0,\ \overline{a_0},\ an_0\rangle)_i\right)\right)\Big/\bigwedge$$

$$\deg_{\text{Poly}[K_0]}\left[\left(\left(\left(\underset{\text{Poly}[K_0]}{x}\ {}_{\text{Poly}[K_0]}\overline{\ }\ \operatorname{const}_{\text{Poly}[K_0]}[x0_0]\right)^*_{\text{Poly}[K_0]}\text{NevillePolynomial}[\langle \overline{x_0},\ xn_0\rangle,\ \langle \overline{a_0},\ an_0\rangle,\ K_0]\ {}_{\text{Poly}[K_0]}\overline{\ }\right.\right.\right.$$
$$\left.\left.\left(\underset{\text{Poly}[K_0]}{x}\ {}_{\text{Poly}[K_0]}\overline{\ }\ \operatorname{const}_{\text{Poly}[K_0]}[xn_0]\right)^*_{\text{Poly}[K_0]}\right.\right.$$
$$\left.\left.\left.\text{NevillePolynomial}[\langle x0_0,\ \overline{x_0}\rangle,\ \langle a0_0,\ \overline{a_0}\rangle,\ K_0]\right)\Big/_{\text{Poly}[K_0]}\left(xn_0\ {}_{\overline{K_0}}\ x0_0\right)\right] \le -1 + (1 + n_0)$$

,

which, using (Definition (Polynomial Domain: knowledge): eval x), is implied by:

$$\forall_{i=1,\ldots,1+n_0}\left(\left(\left((\langle x0_0,\overline{x_0},xn_0\rangle)_i \underset{K_0}{\overline{\ }} x0_0\right) \underset{K_0}{*}\right.\right.$$

$$\operatorname{eval}_{\mathrm{Poly}[K_0]}[\mathrm{NevillePolynomial}[\langle\overline{x_0},xn_0\rangle,\langle\overline{a_0},an_0\rangle,K_0],(\langle x0_0,\overline{x_0},xn_0\rangle)_i]\underset{K_0}{\overline{\ }}$$

$$\left((\langle x0_0,\overline{x_0},xn_0\rangle)_i \underset{K_0}{\overline{\ }} xn_0\right) \underset{K_0}{*} \operatorname{eval}_{\mathrm{Poly}[K_0]}[\mathrm{NevillePolynomial}[\langle x0_0,\overline{x_0}\rangle,\langle a0_0,\overline{a_0}\rangle,K_0],$$

$$\left.\left.(\langle x0_0,\overline{x_0},xn_0\rangle)_i]\right) \Big/_{K_0} \left(xn_0 \underset{K_0}{\overline{\ }} x0_0\right) = (\langle a0_0,\overline{a_0},an_0\rangle)_i\right) \bigwedge$$

$$\deg_{\mathrm{Poly}[K_0]}[\left(\left(\left(x_{\mathrm{Poly}[K_0]} \overline{\mathrm{Poly}[K_0]}\ \mathrm{const}_{\mathrm{Poly}[K_0]}[x0_0]\right) \underset{\mathrm{Poly}[K_0]}{*} \mathrm{NevillePolynomial}[\langle\overline{x_0},xn_0\rangle,\langle\overline{a_0},an_0\rangle,K_0]\right.\right.\underset{\mathrm{Poly}[K_0]}{\overline{\ }}$$

$$\left(x_{\mathrm{Poly}[K_0]} \overline{\mathrm{Poly}[K_0]}\ \mathrm{const}_{\mathrm{Poly}[K_0]}[xn_0]\right) \underset{\mathrm{Poly}[K_0]}{*}$$

$$\left.\left.\mathrm{NevillePolynomial}[\langle x0_0,\overline{x_0}\rangle,\langle a0_0,\overline{a_0}\rangle,K_0]\right) \Big/_{\mathrm{Poly}[K_0]} \left(xn_0 \underset{K_0}{\overline{\ }} x0_0\right)\right] \le -1+(1+n_0)$$

,

which, using ⟨Definition (Polynomial Domain: knowledge): deg /⟩, is implied by:

$$\underset{i=1,\ldots,1+n_0}{\forall}\left(\left(\left((\langle x0_0, \overline{x0_0}, xn_0\rangle)_i \underset{K_0}{\overline{\ }} x0_0\right) \underset{K_0}{*}\right.\right. ,$$

$$\underset{\text{Poly}[K_0]}{\text{eval}}[\text{NevillePolynomial}[\langle \overline{x0_0}, xn_0\rangle, \langle \overline{a0_0}, an_0\rangle, K_0], (\langle x0_0, \overline{x0_0}, xn_0\rangle)_i] \underset{K_0}{\overline{\ }}$$

$$\left((\langle x0_0, \overline{x0_0}, xn_0\rangle)_i \underset{K_0}{\overline{\ }} xn_0\right) \underset{K_0}{*} \underset{\text{Poly}[K_0]}{\text{eval}}[\text{NevillePolynomial}[\langle x0_0, \overline{x0_0}\rangle, \langle a0_0, \overline{a0_0}\rangle, K_0],$$

$$\left.\left.(\langle x0_0, \overline{x0_0}, xn_0\rangle)_i]\right) \Big/ \underset{K_0}{\left(xn_0 \underset{K_0}{\overline{\ }} x0_0\right) = (\langle a0_0, \overline{a0_0}, an_0\rangle)_i}\right) \bigwedge$$

$$\underset{\text{Poly}[K_0]}{\deg}\left[\left(\underset{\text{Poly}[K_0]}{x}\ \underset{\text{Poly}[K_0]}{\overline{\text{Poly}}}\ \underset{\text{Poly}[K_0]}{\text{const}}[x0_0]\right) \underset{\text{Poly}[K_0]}{*} \text{NevillePolynomial}[\langle \overline{x0_0}, xn_0\rangle, \langle \overline{a0_0}, an_0\rangle, K_0] \underset{\text{Poly}[K_0]}{\overline{\text{Poly}}}\right.$$

$$\left(\underset{\text{Poly}[K_0]}{x}\ \underset{\text{Poly}[K_0]}{\overline{\text{Poly}}}\ \underset{\text{Poly}[K_0]}{\text{const}}[xn_0]\right) \underset{\text{Poly}[K_0]}{*}$$

$$\left.\text{NevillePolynomial}[\langle x0_0, \overline{x0_0}\rangle, \langle a0_0, \overline{a0_0}\rangle, K_0]\right] \leq -1 + (1 + n_0)$$

which, using (Definition (Polynomial Domain: knowledge): deg −), is implied by:

$$\underset{i=1,\ldots,1+n_0}{\forall}\left(\left(\left(\left(\langle x0_0, \overline{x_0}, xn_0\rangle\right)_i \underset{\overline{K_0}}{} x0_0\right) \underset{K_0}{*}\right.\right. \qquad ,$$

$$\underset{\text{Poly}[K_0]}{\text{eval}}\left[\text{NevillePolynomial}\left[\langle \overline{x_0}, xn_0\rangle, \langle \overline{a_0}, an_0\rangle, K_0\right], \left(\langle x0_0, \overline{x_0}, xn_0\rangle\right)_i\right] \underset{\overline{K_0}}{}$$

$$\left(\left(\langle x0_0, \overline{x_0}, xn_0\rangle\right)_i \underset{\overline{K_0}}{} xn_0\right) \underset{K_0}{*} \underset{\text{Poly}[K_0]}{\text{eval}}\left[\text{NevillePolynomial}\left[\langle x0_0, \overline{x_0}\rangle, \langle a0_0, \overline{a_0}\rangle, K_0\right],\right.$$

$$\left.\left.\left(\langle x0_0, \overline{x_0}, xn_0\rangle\right)_i\right]\right) \Big/ \underset{K_0}{} \left(xn_0 \underset{\overline{K_0}}{} x0_0\right) = \left(\langle a0_0, \overline{a_0}, an_0\rangle\right)_i \Bigg) \bigwedge$$

$$\underset{\text{Poly}[K_0]}{\deg}\left[\left(\underset{\text{Poly}[K_0]}{x} \underset{\text{Poly}[K_0]}{\overline{\text{Poly}}} \underset{\text{Poly}[K_0]}{\text{const}}\left[x0_0\right]\right) \underset{\text{Poly}[K_0]}{*} \text{NevillePolynomial}\left[\langle \overline{x_0}, xn_0\rangle, \langle \overline{a_0}, an_0\rangle, K_0\right]\right] \leq$$

$$-1 + (1 + n_0) \bigwedge$$

$$\underset{\text{Poly}[K_0]}{\deg}\left[\left(\underset{\text{Poly}[K_0]}{x} \underset{\text{Poly}[K_0]}{\overline{\text{Poly}}} \underset{\text{Poly}[K_0]}{\text{const}}\left[xn_0\right]\right) \underset{\text{Poly}[K_0]}{*} \text{NevillePolynomial}\left[\langle x0_0, \overline{x_0}\rangle, \langle a0_0, \overline{a_0}\rangle, K_0\right]\right] \leq$$

$$-1 + (1 + n_0)$$

which, using (Definition (Polynomial Domain: knowledge): deg x*(.)), is implied by:

(70)

$$\forall_{i=1,\ldots,1+n_0} \left(\left(\left(\left(\langle x0_0, \overline{x_0}, xn_0 \rangle \right)_i \; \overline{K_0} \; x0_0 \right) \underset{K_0}{*} \right. \right. \qquad .$$

$$\operatorname{eval}_{\mathrm{Poly}[K_0]} [\mathrm{NevillePolynomial}[\langle \overline{x_0}, xn_0 \rangle, \langle \overline{a_0}, an_0 \rangle, K_0], (\langle x0_0, \overline{x_0}, xn_0 \rangle)_i] \; \overline{K_0}$$

$$\left(\left(\langle x0_0, \overline{x_0}, xn_0 \rangle \right)_i \; \overline{K_0} \; xn_0 \right) \underset{K_0}{*} \operatorname{eval}_{\mathrm{Poly}[K_0]} [\mathrm{NevillePolynomial}[\langle x0_0, \overline{x_0} \rangle, \langle a0_0, \overline{a_0} \rangle, K_0],$$

$$\left. \left. (\langle x0_0, \overline{x_0}, xn_0 \rangle)_i] \right) \Big/ \underset{K_0}{} \left(xn_0 \; \overline{K_0} \; x0_0 \right) = (\langle a0_0, \overline{a_0}, an_0 \rangle)_i \right) \bigwedge$$

$$\deg_{\mathrm{Poly}[K_0]} [\mathrm{NevillePolynomial}[\langle x0_0, \overline{x_0} \rangle, \langle a0_0, \overline{a_0} \rangle, K_0]] \leq$$

$$-1 + (-1 + (1 + n_0)) \bigwedge$$

$$\deg_{\mathrm{Poly}[K_0]} [\mathrm{NevillePolynomial}[\langle \overline{x_0}, xn_0 \rangle, \langle \overline{a_0}, an_0 \rangle, K_0]] \leq -1 + (-1 + (1 + n_0))$$

Using available computation rules we evaluate (70):

(71)

$$\bigwedge_{i=1,\ldots,1+n_0} \left(\left(\left((\langle x0_0, \overline{x0}, xn_0\rangle)_i \underset{\overline{K_0}}{} x0_0 \right) \underset{K_0}{*} \right. \right.$$

$$\operatorname{eval}_{\operatorname{Poly}[K_0]}[\operatorname{NevillePolynomial}[\langle \overline{x0}, xn_0\rangle, \langle \overline{a0}, an_0\rangle, K_0], (\langle x0_0, \overline{x0}, xn_0\rangle)_i] \underset{\overline{K_0}}{}$$

$$\left((\langle x0_0, \overline{x0}, xn_0\rangle)_i \underset{\overline{K_0}}{} xn_0 \right) \underset{K_0}{*} \operatorname{eval}_{\operatorname{Poly}[K_0]}[\operatorname{NevillePolynomial}[\langle x0_0, \overline{x0}\rangle, \langle a0_0, \overline{a0}\rangle, K_0],$$

$$\left. \left. (\langle x0_0, \overline{x0}, xn_0\rangle)_i] \right) \Big/_{K_0} \left(xn_0 \underset{\overline{K_0}}{} x0_0 \right) = (\langle a0_0, \overline{a0}, an_0\rangle)_i \right) \bigwedge$$

$$\deg_{\operatorname{Poly}[K_0]}[\operatorname{NevillePolynomial}[\langle x0_0, \overline{x0}\rangle, \langle a0_0, \overline{a0}\rangle, K_0]] \leq$$

$$-1 +$$

$$n_0 \bigwedge$$

$$\deg_{\operatorname{Poly}[K_0]}[\operatorname{NevillePolynomial}[\langle \overline{x0}, xn_0\rangle, \langle \overline{a0}, an_0\rangle, K_0]] \leq$$

$$-1 + n_0$$

We prove the individual conjunctive parts of (71):

Proof of (71.1)

$$\bigwedge_{i=1,\ldots,1+n_0} \left(\left(\left((\langle x0_0, \overline{x0}, xn_0\rangle)_i \underset{\overline{K_0}}{} x0_0 \right) \underset{K_0}{*} \right. \right.$$

$$\operatorname{eval}_{\operatorname{Poly}[K_0]}[\operatorname{NevillePolynomial}[\langle \overline{x0}, xn_0\rangle, \langle \overline{a0}, an_0\rangle, K_0], (\langle x0_0, \overline{x0}, xn_0\rangle)_i] \underset{\overline{K_0}}{}$$

$$\left((\langle x0_0, \overline{x0}, xn_0\rangle)_i \underset{\overline{K_0}}{} xn_0 \right) \underset{K_0}{*} \operatorname{eval}_{\operatorname{Poly}[K_0]}[\operatorname{NevillePolynomial}[\langle x0_0, \overline{x0}\rangle, \langle a0_0, \overline{a0}\rangle, K_0],$$

$$\left. \left. (\langle x0_0, \overline{x0}, xn_0\rangle)_i] \right) \Big/_{K_0} \left(xn_0 \underset{\overline{K_0}}{} x0_0 \right) = (\langle a0_0, \overline{a0}, an_0\rangle)_i \right) :$$

For proving (71.1) we distinguish several cases:

Case $i = 1$: In this case the goal (71.1) simplifies to

(72) $\Big(\big(x0_0 \underset{K_0}{\frown} x0_0\big) \underset{K_0}{*} \underset{\text{Poly}[K_0]}{\text{eval}}[\text{NevillePolynomial}[\langle \overline{x_0}, xn_0\rangle, \langle \overline{a_0}, an_0\rangle, K_0], x0_0] \underset{K_0}{\frown} \big(x0_0 \underset{K_0}{\frown} xn_0\big) \underset{K_0}{*} \underset{\text{Poly}[K_0]}{\text{eval}}[\text{NevillePolynomial}[\langle x0_0, \overline{x_0}\rangle, \langle a0_0, \overline{a_0}\rangle, K_0], x0_0]\Big) \Big/ \underset{K_0}{} \big(xn_0 \underset{K_0}{\frown} x0_0\big) = (\langle a0_0, \overline{a_0}, an_0\rangle)_1$.

Using available computation rules we evaluate (72):

(78)

$\Big(\big(x0_0 \underset{K_0}{\frown} x0_0\big) \underset{K_0}{*} \underset{\text{Poly}[K_0]}{\text{eval}}[\text{NevillePolynomial}[\langle \overline{x_0}, xn_0\rangle, \langle \overline{a_0}, an_0\rangle, K_0], x0_0] \underset{K_0}{\frown} \big(x0_0 \underset{K_0}{\frown} xn_0\big) \underset{K_0}{*} \underset{\text{Poly}[K_0]}{\text{eval}}[\text{NevillePolynomial}[\langle x0_0, \overline{x_0}\rangle, \langle a0_0, \overline{a_0}\rangle, K_0], x0_0]\Big) \Big/ \underset{K_0}{} \big(xn_0 \underset{K_0}{\frown} x0_0\big) = a0_0$

Formula (78), using (Definition (Fields: knowledge): Inv +), is implied by:

$\Big(K_0[0] \underset{K_0}{*} \underset{\text{Poly}[K_0]}{\text{eval}}[\text{NevillePolynomial}[\langle \overline{x_0}, xn_0\rangle, \langle \overline{a_0}, an_0\rangle, K_0], x0_0] \underset{K_0}{\frown} \big(x0_0 \underset{K_0}{\frown} xn_0\big) \underset{K_0}{*} \underset{\text{Poly}[K_0]}{\text{eval}}[\text{NevillePolynomial}[\langle x0_0, \overline{x_0}\rangle, \langle a0_0, \overline{a_0}\rangle, K_0], x0_0]\Big) \Big/ \underset{K_0}{} \big(xn_0 \underset{K_0}{\frown} x0_0\big) = a0_0$,

which, using (Definition (Fields: knowledge): 0 *), is implied by:

$\Big(K_0[0] \underset{K_0}{\frown} \big(x0_0 \underset{K_0}{\frown} xn_0\big) \underset{K_0}{*} \underset{\text{Poly}[K_0]}{\text{eval}}[\text{NevillePolynomial}[\langle x0_0, \overline{x_0}\rangle, \langle a0_0, \overline{a_0}\rangle, K_0], x0_0]\Big) \Big/ \underset{K_0}{} \big(xn_0 \underset{K_0}{\frown} x0_0\big) = a0_0$,

which, using (Definition (Fields: knowledge): Neut +), is implied by:

$\Big(\underset{K_0}{\frown}\big(x0_0 \underset{K_0}{\frown} xn_0\big) \underset{K_0}{*} \underset{\text{Poly}[K_0]}{\text{eval}}[\text{NevillePolynomial}[\langle x0_0, \overline{x_0}\rangle, \langle a0_0, \overline{a_0}\rangle, K_0], x0_0]\Big) \Big/ \underset{K_0}{} \big(xn_0 \underset{K_0}{\frown} x0_0\big) = a0_0$,

which, using (Definition (Fields: knowledge): Inv * neg), is implied by:

(79) $\underset{\text{Poly}[K_0]}{\text{eval}}[\text{NevillePolynomial}[\langle x0_0, \overline{x_0}\rangle, \langle a0_0, \overline{a_0}\rangle, K_0], x0_0] = a0_0$.

Using equalities in the knowledge base, the goal (79) simplifies to

(83) $(\langle a0_0, \overline{a_0}\rangle)_1 = a0_0$.

Using available computation rules we evaluate (83):

(84) True .

Formula (84) is true because it is the constant True.

Case $2 \le i \le n_0$: We assume

(73) IsInteger$[i_0]$,

(74) $i_0 \ge 2$,

(75) $i_0 \le n_0$

and prove

(76)

$$\Big(\big((\langle x0_0, \overline{x_0}, xn_0\rangle)_{i_0}\ \overline{K_0}\ x0_0\big) \underset{K_0}{*} \underset{\text{Poly}[K_0]}{\text{eval}}[\text{NevillePolynomial}[\langle \overline{x_0}, xn_0\rangle, \langle \overline{a_0}, an_0\rangle, K_0], (\langle x0_0, \overline{x_0}, xn_0\rangle)_{i_0}]\ \overline{K_0}$$
$$\big((\langle x0_0, \overline{x_0}, xn_0\rangle)_{i_0}\ \overline{K_0}\ xn_0\big) \underset{K_0}{*} \underset{\text{Poly}[K_0]}{\text{eval}}[\text{NevillePolynomial}[\langle x0_0, \overline{x_0}\rangle, \langle a0_0, \overline{a_0}\rangle, K_0],$$
$$(\langle x0_0, \overline{x_0}, xn_0\rangle)_{i_0}]\Big) \Big/ \underset{K_0}{} \big(xn_0\ \overline{K_0}\ x0_0\big) = (\langle a0_0, \overline{a_0}, an_0\rangle)_{i_0}$$

We normalize all tuples in the goal (76). Hence, it remains to show

(91)

$$\Big(\big((\langle \overline{x_0}\rangle)_{-1+i_0}\ \overline{K_0}\ x0_0\big) \underset{K_0}{*} \underset{\text{Poly}[K_0]}{\text{eval}}[\text{NevillePolynomial}[\langle \overline{x_0}, xn_0\rangle, \langle \overline{a_0}, an_0\rangle, K_0], (\langle \overline{x_0}\rangle)_{-1+i_0}]\ \overline{K_0}\ .$$
$$\big((\langle \overline{x_0}\rangle)_{-1+i_0}\ \overline{K_0}\ xn_0\big) \underset{K_0}{*} \underset{\text{Poly}[K_0]}{\text{eval}}[\text{NevillePolynomial}[\langle x0_0, \overline{x_0}\rangle, \langle a0_0, \overline{a_0}\rangle, K_0],$$
$$(\langle \overline{x_0}\rangle)_{-1+i_0}]\Big) \Big/ \underset{K_0}{} \big(xn_0\ \overline{K_0}\ x0_0\big) = (\langle \overline{a_0}\rangle)_{-1+i_0}$$

Using equalities in the knowledge base, the goal (91) simplifies to

(97)
$$\Big(\big((\langle \overline{x_0}\rangle)_{-1+i_0}\ \overline{K_0}\ x0_0\big) \underset{K_0}{*} (\langle \overline{a_0}\rangle)_{-1+i_0}\ \overline{K_0}\ .$$
$$\big((\langle \overline{x_0}\rangle)_{-1+i_0}\ \overline{K_0}\ xn_0\big) \underset{K_0}{*} \underset{\text{Poly}[K_0]}{\text{eval}}[\text{NevillePolynomial}[\langle x0_0, \overline{x_0}\rangle, \langle a0_0, \overline{a_0}\rangle, K_0],$$
$$(\langle \overline{x_0}\rangle)_{-1+i_0}]\Big) \Big/ \underset{K_0}{} \big(xn_0\ \overline{K_0}\ x0_0\big) = (\langle \overline{a_0}\rangle)_{-1+i_0}$$

Using equalities in the knowledge base, the goal (97) simplifies to

(103)

$$\Big(\big((\langle \overline{x_0}\rangle)_{-1+i_0}\ \overline{K_0}\ x0_0\big) \underset{K_0}{*} (\langle \overline{a_0}\rangle)_{-1+i_0}\ \overline{K_0}\ \big((\langle \overline{x_0}\rangle)_{-1+i_0}\ \overline{K_0}\ xn_0\big) \underset{K_0}{*} (\langle \overline{a_0}\rangle)_{-1+i_0}\Big) \Big/ \underset{K_0}{} \big(xn_0\ \overline{K_0}\ x0_0\big) = .$$
$$(\langle \overline{a_0}\rangle)_{-1+i_0}$$

Formula (103), using (Definition (Fields: knowledge): Dist –), is implied by:

$$\left(\left(\left(\left(\langle\overline{x_0}\rangle\right)_{-1+i_0}\ \underset{K_0}{\frown}\ x0_0\right)\underset{K_0}{\frown}\left(\left(\langle\overline{x_0}\rangle\right)_{-1+i_0}\ \underset{K_0}{\frown}\ xn_0\right)\right)\underset{K_0}{*}\left(\langle\overline{a_0}\rangle\right)_{-1+i_0}\right)\Big/\underset{K_0}{}\left(xn_0\ \underset{K_0}{\frown}\ x0_0\right)=\left(\langle\overline{a_0}\rangle\right)_{-1+i_0},$$

which, using (Definition (Fields: knowledge): −(.)), is implied by:

$$\left(\left(xn_0\ \underset{K_0}{\frown}\ x0_0\right)\underset{K_0}{*}\left(\langle\overline{a_0}\rangle\right)_{-1+i_0}\right)\Big/\underset{K_0}{}\left(xn_0\ \underset{K_0}{\frown}\ x0_0\right)=\left(\langle\overline{a_0}\rangle\right)_{-1+i_0},$$

which, using (Definition (Fields: knowledge): Inv *), is implied by:

(104) $\left(\langle\overline{a_0}\rangle\right)_{-1+i_0}=\left(\langle\overline{a_0}\rangle\right)_{-1+i_0}$.

Using available computation rules we evaluate (104):

(105) True.

Formula (105) is true because it is the constant True.

Case $i = 1 + n_0$: In this case the goal (71.1) simplifies to

(77) $$\left(\left(xn_0\ \underset{K_0}{\frown}\ x0_0\right)\underset{K_0}{*}\underset{\text{Poly}[K_0]}{\text{eval}}[\text{NevillePolynomial}[\langle\overline{x_0}, xn_0\rangle, \langle\overline{a_0}, an_0\rangle, K_0], xn_0]\ \underset{K_0}{\frown}\right.$$

$$\left.\left(xn_0\ \underset{K_0}{\frown}\ xn_0\right)\underset{K_0}{*}\underset{\text{Poly}[K_0]}{\text{eval}}[\text{NevillePolynomial}[\langle x0_0, \overline{x_0}\rangle, \langle a0_0, \overline{a_0}\rangle, K_0], xn_0]\right)\Big/\underset{K_0}{}$$

$$\left(xn_0\ \underset{K_0}{\frown}\ x0_0\right)=\left(\langle a0_0, \overline{a_0}, an_0\rangle\right)_{1+n_0}$$

We normalize all tuples in the goal (77). Hence, it remains to show

(106) $$\left(\left(xn_0\ \underset{K_0}{\frown}\ x0_0\right)\underset{K_0}{*}\underset{\text{Poly}[K_0]}{\text{eval}}[\text{NevillePolynomial}[\langle\overline{x_0}, xn_0\rangle, \langle\overline{a_0}, an_0\rangle, K_0], xn_0]\ \underset{K_0}{\frown}\right.$$

$$\left.\left(xn_0\ \underset{K_0}{\frown}\ xn_0\right)\underset{K_0}{*}\underset{\text{Poly}[K_0]}{\text{eval}}[\text{NevillePolynomial}[\langle x0_0, \overline{x_0}\rangle, \langle a0_0, \overline{a_0}\rangle, K_0], xn_0]\right)\Big/\underset{K_0}{}$$

$$\left(xn_0\ \underset{K_0}{\frown}\ x0_0\right)=\left(\langle\overline{a_0}, an_0\rangle\right)_{n_0}$$

Formula (106), using (Definition (Fields: knowledge): Inv +), is implied by:

$$\left(\left(xn_0\ \underset{K_0}{\frown}\ x0_0\right)\underset{K_0}{*}\underset{\text{Poly}[K_0]}{\text{eval}}[\text{NevillePolynomial}[\langle\overline{x_0}, xn_0\rangle, \langle\overline{a_0}, an_0\rangle, K_0], xn_0]\ \underset{K_0}{\frown}\right.,$$

$$\left.K_0[0]\underset{K_0}{*}\underset{\text{Poly}[K_0]}{\text{eval}}[\text{NevillePolynomial}[\langle x0_0, \overline{x_0}\rangle, \langle a0_0, \overline{a_0}\rangle, K_0], xn_0]\right)\Big/\underset{K_0}{}$$

$$\left(xn_0\ \underset{K_0}{\frown}\ x0_0\right)=\left(\langle\overline{a_0}, an_0\rangle\right)_{n_0}$$

which, using (Definition (Fields: knowledge): 0 *), is implied by:

$$\left(\left(xn_0\ \underset{K_0}{\frown}\ x0_0\right)\underset{K_0}{*}\underset{\text{Poly}[K_0]}{\text{eval}}[\text{NevillePolynomial}[\langle\overline{x_0}, xn_0\rangle, \langle\overline{a_0}, an_0\rangle, K_0], xn_0]\ \underset{K_0}{\frown}\ K_0[0]\right)\Big/\underset{K_0}{},$$

$$\left(xn_0\ \underset{K_0}{\frown}\ x0_0\right)=\left(\langle\overline{a_0}, an_0\rangle\right)_{n_0}$$

which, using (Definition (Fields: knowledge): Neut −), is implied by:

$$\left(\left(xn_0\ \underset{K_0}{\frown}\ x0_0\right)\underset{K_0}{*}\underset{\text{Poly}[K_0]}{\text{eval}}[\text{NevillePolynomial}[\langle\overline{x_0}, xn_0\rangle, \langle\overline{a_0}, an_0\rangle, K_0], xn_0]\right)\Big/\underset{K_0}{}\left(xn_0\ \underset{K_0}{\frown}\ x0_0\right)=,$$

$$\left(\langle\overline{a_0}, an_0\rangle\right)_{n_0}$$

which, using (Definition (Fields: knowledge): Inv *), is implied by:

(107) $\underset{\text{Poly}[K_0]}{\text{eval}}[\text{NevillePolynomial}[\langle\overline{x_0}, xn_0\rangle, \langle\overline{a_0}, an_0\rangle, K_0], xn_0]=\left(\langle\overline{a_0}, an_0\rangle\right)_{n_0}$.

Formula (58), by (Definition (Interpolating polynomial: characterization)), implies:

$$\forall_{i=1,\ldots,|\langle \overline{x_0}, xn_0\rangle|} \left(\operatorname{eval}_{\text{Poly}[K_0]} [\text{NevillePolynomial}[\langle \overline{x_0}, xn_0\rangle, \langle \overline{a_0}, an_0\rangle, K_0], (\langle \overline{x_0}, xn_0\rangle)_i] = \right. ,$$

$$\left. (\langle \overline{a_0}, an_0\rangle)_i \right) \bigwedge \deg_{\text{Poly}[K_0]} [\text{NevillePolynomial}[\langle \overline{x_0}, xn_0\rangle, \langle \overline{a_0}, an_0\rangle, K_0]] \leq -1 + |\langle \overline{x_0}, xn_0\rangle|$$

which, by (30), implies:

(108)

$$\forall_{i=1,\ldots,n_0} \left(\operatorname{eval}_{\text{Poly}[K_0]} [\text{NevillePolynomial}[\langle \overline{x_0}, xn_0\rangle, \langle \overline{a_0}, an_0\rangle, K_0], (\langle \overline{x_0}, xn_0\rangle)_i] = (\langle \overline{a_0}, an_0\rangle)_i \right) \bigwedge .$$

$$\deg_{\text{Poly}[K_0]} [\text{NevillePolynomial}[\langle \overline{x_0}, xn_0\rangle, \langle \overline{a_0}, an_0\rangle, K_0]] \leq -1 + n_0$$

Assumption (108.1) specializes to

(109)

$$\forall_{i=1,\ldots,-1+n_0} \left(\operatorname{eval}_{\text{Poly}[K_0]} [\text{NevillePolynomial}[\langle \overline{x_0}, xn_0\rangle, \langle \overline{a_0}, an_0\rangle, K_0], (\langle \overline{x_0}, xn_0\rangle)_i] = (\langle \overline{a_0}, an_0\rangle)_i \right),$$

(110) $$\operatorname{eval}_{\text{Poly}[K_0]} [\text{NevillePolynomial}[\langle \overline{x_0}, xn_0\rangle, \langle \overline{a_0}, an_0\rangle, K_0], xn_0] = (\langle \overline{a_0}, an_0\rangle)_{n_0} .$$

Formula (107) is true because it is identical to (110).

Proof of (71.2) $\deg_{\text{Poly}[K_0]} [\text{NevillePolynomial}[\langle x0_0, \overline{x_0}\rangle, \langle a0_0, \overline{a_0}\rangle, K_0]] \leq -1 + n_0$:

Formula (65), by (Definition (Interpolating polynomial: characterization)), implies:

$$\forall_{i=1,\ldots,|\langle x0_0, \overline{x_0}\rangle|} \left(\operatorname{eval}_{\text{Poly}[K_0]} [\text{NevillePolynomial}[\langle x0_0, \overline{x_0}\rangle, \langle a0_0, \overline{a_0}\rangle, K_0], (\langle x0_0, \overline{x_0}\rangle)_i] = \right. ,$$

$$\left. (\langle a0_0, \overline{a_0}\rangle)_i \right) \bigwedge \deg_{\text{Poly}[K_0]} [\text{NevillePolynomial}[\langle x0_0, \overline{x_0}\rangle, \langle a0_0, \overline{a_0}\rangle, K_0]] \leq -1 + |\langle x0_0, \overline{x_0}\rangle|$$

which, by (31), implies:

(111)

$$\forall_{i=1,\ldots,n_0} \left(\operatorname{eval}_{\text{Poly}[K_0]} [\text{NevillePolynomial}[\langle x0_0, \overline{x_0}\rangle, \langle a0_0, \overline{a_0}\rangle, K_0], (\langle x0_0, \overline{x_0}\rangle)_i] = (\langle a0_0, \overline{a_0}\rangle)_i \right) \bigwedge .$$

$$\deg_{\text{Poly}[K_0]} [\text{NevillePolynomial}[\langle x0_0, \overline{x_0}\rangle, \langle a0_0, \overline{a_0}\rangle, K_0]] \leq -1 + n_0$$

Formula (71.2) is true because it is identical to (111.2).

Proof of (71.3) $\deg_{\text{Poly}[K_0]} [\text{NevillePolynomial}[\langle \overline{x_0}, xn_0\rangle, \langle \overline{a_0}, an_0\rangle, K_0]] \leq -1 + n_0$:

Formula (58), by (Definition (Interpolating polynomial: characterization)), implies:

$$\forall_{i=1,\ldots,|\langle \overline{x_0}, xn_0\rangle|} \left(\operatorname{eval}_{\text{Poly}[K_0]} [\text{NevillePolynomial}[\langle \overline{x_0}, xn_0\rangle, \langle \overline{a_0}, an_0\rangle, K_0], (\langle \overline{x_0}, xn_0\rangle)_i] = \right. ,$$

$$\left. (\langle \overline{a_0}, an_0\rangle)_i \right) \bigwedge \deg_{\text{Poly}[K_0]} [\text{NevillePolynomial}[\langle \overline{x_0}, xn_0\rangle, \langle \overline{a_0}, an_0\rangle, K_0]] \leq -1 + |\langle \overline{x_0}, xn_0\rangle|$$

which, by (30), implies:

(112)

$$\forall_{i=1,\ldots,n_0} \left(\operatorname{eval}_{\text{Poly}[K_0]} [\text{NevillePolynomial}[\langle \overline{x_0}, xn_0\rangle, \langle \overline{a_0}, an_0\rangle, K_0], (\langle \overline{x_0}, xn_0\rangle)_i] = (\langle \overline{a_0}, an_0\rangle)_i \right) \bigwedge .$$

$$\deg_{\text{Poly}[K_0]} [\text{NevillePolynomial}[\langle \overline{x_0}, xn_0\rangle, \langle \overline{a_0}, an_0\rangle, K_0]] \leq -1 + n_0$$

Formula (71.3) is true because it is identical to (112.2).

□